LA 10.6.2013 Meno:

zápočet	Pr1	Pr2	Pr3	Pr4	Pr5	Pr6	Pr7	$\sum$	známka

v poliach R a C

1. [13 bodov] Daná je matica $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ -2 & 3 & 4 \end{pmatrix}$ a $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Určte

   a) [1] stopu matice A, b) [3] minimálny polynóm $m_{A,\mathbf{b}}(\lambda)$ (vektora $\mathbf{b}$ vzhľadom na maticu A),

   c) [3] vlastné čísla matice A, d) [4] Jordanovu maticu J a maticu P, pre ktorú $A = PJP^{-1}$,

   e) [2] maticu $f(J)$, pre $f(\lambda) = (\lambda - 2)^9 + \lambda^2 - 1$.

2. [8] Nájdite rovnicu priamky $y(x) = kx + q$, pre ktorú je súčet $\sum_{i=-2}^{2} |y(x_i) - y_i|^2$ najmenší. Hodnoty y_i sú v tabuľke →

x_i	-2	-1	0	1	2
y_i	-1	0	2	3	4

3. [10] Nech $A = \begin{pmatrix} 2 & -2 & -1 & 0 & 2 \\ 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 6 & 3 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix} \in R^{4\times 5}$. $L_r(A) = \mathrm{span}\{A_{1*}, A_{2*}, A_{3*}, A_{4*}\}$ je riadkový, $L_s(A) = \mathrm{span}\{A_{*1}, A_{*2}, A_{*3}, A_{*4}, A_{*5}\}$ stĺpcový a $N(A) = \{\mathbf{x} \in R^{5\times 1} \colon A\mathbf{x} = 0\}$ nulový priestor matice A. Určte bázy a dimenzie priestorov $L_r(A)$, $L_s(A)$ a $N(A)$.

4. Nech L, M sú lineárne priestory nad poľom R, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}$ je báza priestoru L, $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ je báza priestoru M. Napíšte

   a) (1 bod) $\dim M$ b) (1 bod) súradnice $\left[2\mathbf{b}_1 - \mathbf{b}_3 + 3\mathbf{b}_4\right]_{\mathcal{B}}$

   c) [3] maticu $T_{\mathcal{BD}}$ lineárneho operátora T určeného vzťahmi
   $T\mathbf{b}_1 = \mathbf{d}_2$, $T\mathbf{b}_2 = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$, $T\mathbf{b}_3 = 2\mathbf{d}_1 + \mathbf{d}_2 + 2\mathbf{d}_3$, $T\mathbf{b}_4 = 0$, $T\mathbf{b}_5 = -\mathbf{d}_1 + 2\mathbf{d}_2 - \mathbf{d}_3$.

   d) [3] $\dim\ker T$ a $\dim\mathrm{ran}\, T$

V konečných poliach

5. [6 bodov] Pomocou Euklidovho algoritmu zistite, či má polynóm $f(x) = x^4 + x^3 + x + 1 \in P(Z_3)$ koreň násobnosti aspoň 2.

6. [10] V poli F riešte sústavu lineárnych rovníc

   a) $F = Z_2$
$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ x_2 + x_3 &= 0 \\ x_1 + x_2 + x_4 &= 1 \end{aligned}$$

   b) $F = Z_3$
$$\begin{aligned} x_1 + 2x_3 + x_4 &= 1 \\ 2x_2 + x_3 &= 2 \\ x_1 + x_2 + 2x_4 &= 1 \end{aligned}$$

7. a) [2] Napíšte koľko prvkov má pole $F = P(Z_2)/(x^3 + x + 1)$.

   b) [3] Vypočítajte $(x+1)(x^2+x) \pmod{x^3 + x + 1}$.

   c* [3] Dokážte tvrdenie: Ak je $f(x) \in P(F)$ ireducibilný polynóm, tak má každý prvok $0 \neq h \in P(F)/f(x)$ inverzný (t.j. $\exists g \in P(F)$, pre ktoré $g(x)h(x) = 1 \pmod{f(x)}$).

Riešenie.

1a. $\operatorname{trace}(A) = 6$,

b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ -2 & 3 & 4 \end{pmatrix}$, $Ab = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $A^2b = A(Ab) = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$, $A^3b = A(A^2b) = \begin{pmatrix} 20 \\ 8 \\ 12 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 3 & 8 & 20 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 - R_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 0 & 1 & 4 & 12 \end{pmatrix}, \quad \begin{matrix} a_3 = 0, a_2 = 1 \\ a_1 = -4 \\ a_0 = 4 \end{matrix}, \text{ t.j. } m_{A,\mathbf{b}}(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

c) $\lambda_{1,2} = 2$, $2 + 2 + \lambda_3 = \operatorname{trace}(A) = 6 \implies \lambda_{1,2,3} = 2$

d) Z rovnice $(A - 2I)P_{*3} = 0$ dostaneme $P_{*3} = (1 \quad 0 \quad 1)^\top$, potom riešime sústavy $(A - 2I)P_{*2} = P_{*3}$ a $(A - 2I)P_{*1} = P_{*2}$:

$$\left(\begin{array}{rrr|rr} -1 & 2 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 & 1 \\ -2 & 3 & 2 & 1 & 0 \end{array}\right) \begin{matrix} R_1 + R_2 \\ R_3 + 2R_2 \end{matrix} \sim \left(\begin{array}{rrr|rr} 0 & 1 & 0 & 1 & 2 \\ 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right), \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \; J = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

e) $f(2) = 3$; $f'(\lambda) = 9(\lambda - 2)^8 + 2\lambda$, $f'(2) = 4$; $f''(\lambda) = 72(\lambda - 2)^7 + 2$, $\frac{f''(2)}{2!} = 1 \implies f(J) = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 4 & 3 \end{pmatrix}$

2. $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. $A^\top A = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$, $A^\top \mathbf{b} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$.

$A^\top A \begin{pmatrix} k \\ q \end{pmatrix} = A^\top \mathbf{b} \implies k = 1,3\,,\ q = 1,6$, $\underline{y(x) = 1,3x + 1,6}$.

3.

$$\begin{pmatrix} 2 & -2 & -1 & 0 & 2 \\ 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 6 & 3 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} \underline{1} & -1 & 1 & 0 & 2 \\ 0 & 0 & \underline{3} & 0 & 2 \\ 0 & 0 & 0 & \underline{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$$

$\mathcal{B}_r = \{B_{1*}, B_{2*}, B_{3*}\} = \{(1, -1, 1, 0, 2); (0, 0, 3, 0, 2); (0, 0, 0, 1, -1)\}$, $\dim L_r(A) = 3 = \dim L_s(A)$
$\mathcal{B}_s = \{A_{1*}, A_{3*}, A_{4*}\}$ (stĺpce matice A, v ktorých má upravená matica pivoty), $\dim N(A) = 5 - 3 = 2$,
$\mathcal{B}_{N(A)} = \{(-4, 0, -2, 3, 3)^\top; (1, 1, 0, 0, 0)^\top\}$ (báza riešenia sústavy $Ax = 0$).

4. a) $\dim M = 3$, b) $[2\mathbf{b}_1 - \mathbf{b}_3 + 3\mathbf{b}_4]_{\mathcal{B}} = (2, 0, -1, 3, 0)^\top$, c) $T_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 0 & 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \end{pmatrix}$,

d) $\dim \ker T = 3$, $\dim \operatorname{ran} T = 2$.

5. $f_1(x) = x^4 + x_3 + x + 1$, $f_2(x) = f'(x) = x^3 + 1$
$(x^4 + x_3 + x + 1) : (x^3 + 1) = x + 1 \implies \gcd(f, f') = x^3 + 1$, $c = -1$ je koreň polynómu $f' \implies$ polynóm f má koreň násobnosti ≥ 2.

6a. $P = \{(1, 0, 0, 0); (0, 0, 0, 1); (0, 1, 1, 0); (1, 1, 1, 1)\}$

6b. $P = \{(2, 2, 1, 0); (2, 0, 2, 1); (2, 1, 0, 2)\}$

7a. $2^3 = 8$

7b. $(x + 1)(x^2 + x) = x(x + 1)^2 = x^3 + x = (x^3 + x + 1) + 1 = 1 \pmod{x^3 + x + 1}$ (v $P(Z_2)$)

7c* $h \neq 0 \pmod{f(X)}, \operatorname{stup} h < \operatorname{stup} f \implies \gcd(h, f) = 1$.
Teda $\exists a(x), b(x)$, pre ktoré $1 = a(x)h(x) + b(x)f(x) = a(x)h(x) \pmod{f(x)}$, t.j. $g(x) = a(x)$.