Difference Equations
to
Differential Equations

Section 1.5

Nonlinear Difference Equations

In Section 1.4 we discussed the difference equation

$$x_{n+1} = \alpha x_n, \tag{1.5.1}$$

$n = 0, 1, 2, \ldots$, as a model for either growth or decay and we saw that its solution is given by

$$x_n = \alpha^n x_0,$$

$n = 0, 1, 2, \ldots$. Now

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0, & 0 < \alpha < 1, \\ 1, & \alpha = 1, \\ \infty, & \alpha > 1, \end{cases} \tag{1.5.2}$$

from which it follows that if $\{x_n\}$ is a solution of (1.5.1) with $x_0 > 0$, then

$$\lim_{n\to\infty} x_n = x_0 \lim_{n\to\infty} \alpha^n = \begin{cases} 0, & 0 < \alpha < 1, \\ x_0, & \alpha = 1, \\ \infty, & \alpha > 1. \end{cases} \tag{1.5.3}$$

These limiting values are consistent with our radioactive decay example since, in that case, $0 < \alpha < 1$ and we would expect the amount of a radioactive element to decline toward 0 over time. The case $0 < \alpha < 1$ also may make sense for a population model if the population is declining and heading toward extinction. However, the unbounded growth indefinitely into the future implied by the case $\alpha > 1$ is very unlikely for a population model: eventually ecological or even sociological problems come to the forefront, such as when the population begins to overreach the resources available to it, and the rate of growth of the population changes. Even for bacteria growing in a Petri dish, diminishing food and space eventually cause a change in the rate of growth. Hence the equation

$$x_{n+1} = \alpha x_n, \tag{1.5.4}$$

for $n = 0, 1, 2, \ldots$ and $\alpha > 1$, called the *uninhibited*, or *natural*, *growth model*, although often accurate as a model of population growth over short periods of time, is usually too simplistic for predictions over long time spans.

The inhibited growth model

Suppose we wish to model the growth of a certain population which, without ecological constraints, would grow at a rate of $100\beta\%$ per unit of time. That is, if x_n represents the size of the population after n units of time and there are no constraints on the size of the population, then

$$x_{n+1} - x_n = \beta x_n \tag{1.5.5}$$

for $n = 0, 1, 2, \ldots$. However, suppose that, because of the limitation of resources, the population will begin to decline if it ever has more than M individuals. We call M the *carrying capacity* of the available resources, the maximum population which is sustainable over time. Then it would be reasonable to modify our model by forcing the amount of increase over a unit of time to decrease as the size of the population approaches M and to become negative if the size of the population ever exceeds M. One way to accomplish this is to multiply the term βx_n in (1.5.5) by

$$\frac{M - x_n}{M},$$

a ratio which is close to 1 when x_n is small, close to 0 when x_n is close to M, and negative when x_n exceeds M. This leads us to the difference equation

$$x_{n+1} - x_n = \beta x_n \left(\frac{M - x_n}{M} \right),$$

$n = 0, 1, 2, \ldots$, or, equivalently,

$$x_{n+1} = x_n + \frac{\beta}{M} x_n (M - x_n), \tag{1.5.6}$$

$n = 0, 1, 2, \ldots$, which we call the *inhibited growth model*, also known as the *discrete logistic equation*. This is an example of a nonlinear difference equation because if we multiply out the right-hand side of the equation we have a quadratic term, namely, $\frac{\beta}{M} x_n^2$. Such equations are, in general, far more difficult to solve than the linear difference equations we considered in Section 1.4; in fact, many nonlinear difference equations are not solvable in terms of the elementary functions of calculus. Hence we will not consider any methods for solving such equations, relying instead on computing specific solutions by iterating the equation using a calculator or, preferably, a computer.

Example Suppose a population of owls, currently numbering 100, has a natural growth rate of 4%, but, because of the limited resources of their natural habitat, can sustain a population of no more than 500. If we let x_n represent the size of the population n years from now, then, using the inhibited growth model, we should have

$$x_{n+1} = x_n + \frac{0.04}{500} x_n (500 - x_n) = x_n + 0.00008 x_n (500 - x_n)$$

for $n = 0, 1, 2, \ldots$. Using this equation we are able to compute, for example, the predicted size of the population for the next 10 years:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	103.2	106.5	109.8	113.3	116.8	120.3	124.0	127.8	131.5	135.4

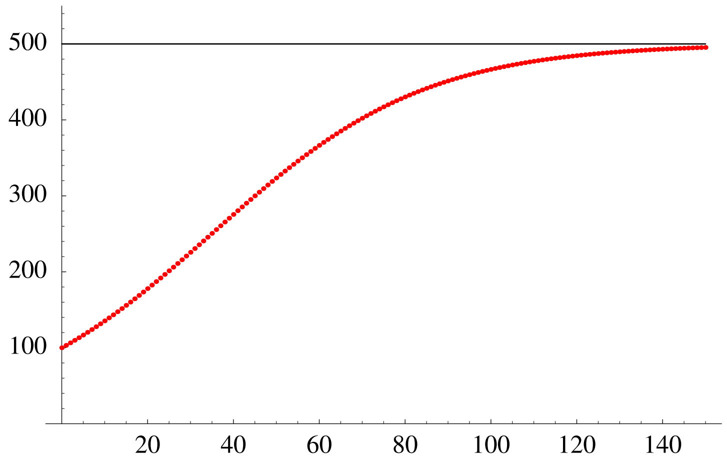

Figure 1.5.1 Inhibited population growth with $\beta = 0.04$

Here, and in subsequent tables, we have rounded our results to the first decimal place. It is interesting to compare these results to the corresponding results for the uninhibited growth model. If we let y_n be the predicted population n years from now using the uninhibited growth model, then we would have

$$y_{n+1} = y_n + 0.04y_n = 1.04y_n,$$

$n = 0, 1, 2, \ldots$, which has the exact solution

$$y_n = 100(1.04)^n$$

for $n = 0, 1, 2, \ldots$. From this model we obtain the following predicted population sizes:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	104.0	108.2	112.5	117.0	121.7	126.5	131.6	136.9	142.3	148.0

As we would expect, the population is growing more slowly under the inhibited population growth model than under the uninhibited model. Moreover, this difference will become more pronounced over time. For example, after 150 years we would have $x_{150} = 495.4$ and $y_{150} = 35,892$, illustrating how the inhibited growth model is constrained by the carrying capacity of 500 while the uninhibited growth model will have unbounded growth. Figures 1.5.1 and 1.5.2 provide a graphical comparison of the two models for $n = 0, 1, 2, \ldots, 150$. Note that it appears that

$$\lim_{n\to\infty} x_n = 500,$$

while

$$\lim_{n\to\infty} y_n = \infty.$$

With the inhibited growth model, if $0 < \beta < 1$ and $x_n < M$, then

$$\beta\frac{x_n}{M} < 1,$$

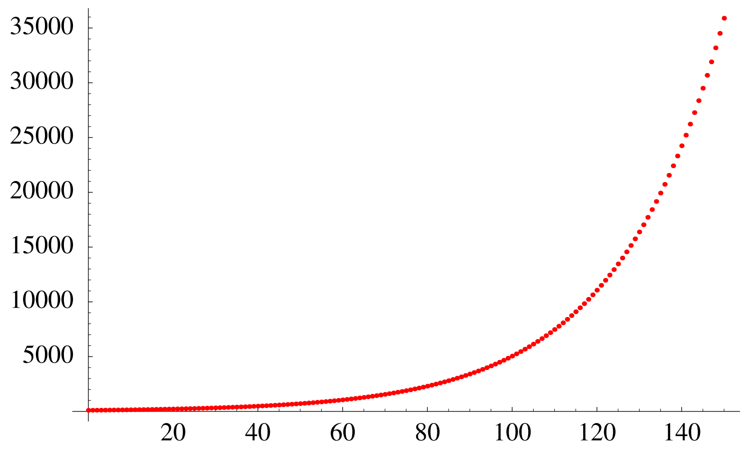

Figure 1.5.2 Uninhibited population growth with $\beta = 0.04$

so

$$x_{n+1} = x_n + \beta\frac{x_n}{M}(M - x_n) < x_n + (M - x_n) = M \tag{1.5.7}$$

for $n = 0, 1, 2, \ldots$. Thus if $0 < \beta < 1$, and we start with $x_0 < M$, then $x_n < M$ for all n. Moreover, since

$$\beta\frac{x_n}{M} > 0,$$

we have

$$x_{n+1} = x_n + \beta\frac{x_n}{M}(M - x_n) > x_n$$

for all n. Hence the sequence $\{x_n\}$ is monotone increasing and bounded, and so must have a limit. In Problem 8 you will be asked to verify that this limit is in fact M, as appeared to be the case in the previous example.

If $\beta > 1$, it may be the case that there are values of n for which $x_n > M$, in which case

$$\beta\frac{x_n}{M}(M - x_n) < 0$$

and, as a consequence, $x_{n+1} < x_n$.

Example Suppose $x_0 = 100$ and $M = 500$ as in the previous example, but now let $\beta = 1.5$. That is,

$$x_{n+1} = x_n + \frac{1.5}{500}x_n(M - x_n) = x_n + 0.003x_n(500 - x_n)$$

for $n = 0, 1, 2, \ldots$. This equation generates the following values:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	220.0	404.8	520.4	448.5	505.3	497.2	501.4	499.3	500.3	499.8

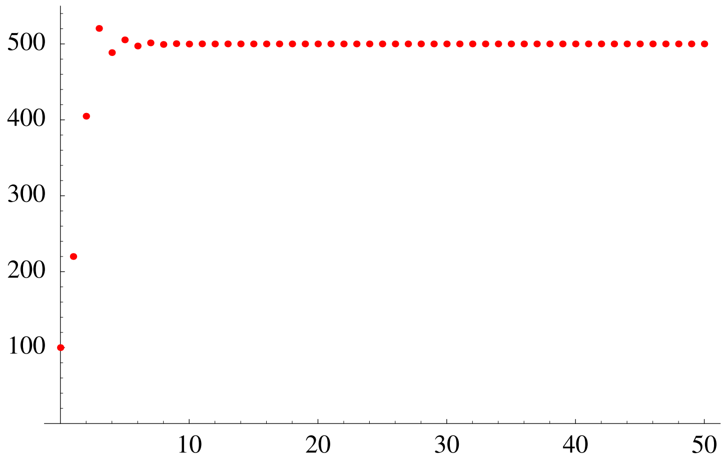

Figure 1.5.3 Inhibited population growth with $\beta = 1.5$

Note how the values increase rapidly (as we should expect with such a large value for β) to above the carrying capacity of 500, but then oscillate about 500, with the oscillations diminishing in size. In fact it may be shown that it is also true in this case that

$$\lim_{n\to\infty} x_n = 500.$$

See Figure 1.5.3.

It is possible to show that, for the inhibited growth model of (1.5.6),

$$\lim_{n\to\infty} x_n = M$$

whenever $0 < \beta \leq 2$. However, there are other possible behaviors when $\beta > 2$.

Example Suppose $x_0 = 100$ and $M = 500$, as in the previous examples, but now let $\beta = 2.3$. That is,

$$x_{n+1} = x_n + \frac{2.3}{500}x_n(M - x_n) = x_n + 0.0046x_n(500 - x_n)$$

for $n = 0, 1, 2, \ldots$. This equation generates the following values:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	284.0	566.2	393.8	586.2	353.8	591.7	342.0	590.6	344.5	590.9

Year	11	12	13	14	15	16	17	18	19	20
Population	343.8	590.8	344.0	590.9	343.9	590.8	343.9	590.8	343.9	590.8

Notice that instead of approaching a single limiting value, the population is settling down to an oscillation between 344 and 591. We say that the sequence $\{x_n\}$ is approaching a *limiting cycle of period 2*, as shown in Figure 1.5.4.

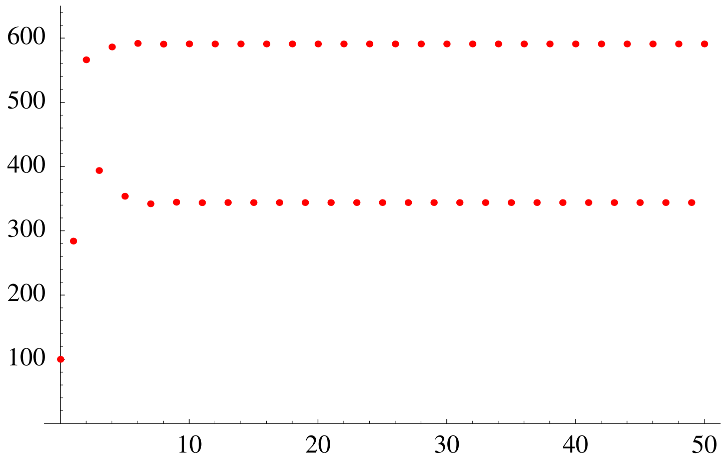

Figure 1.5.4 Inhibited population growth with $\beta = 2.3$

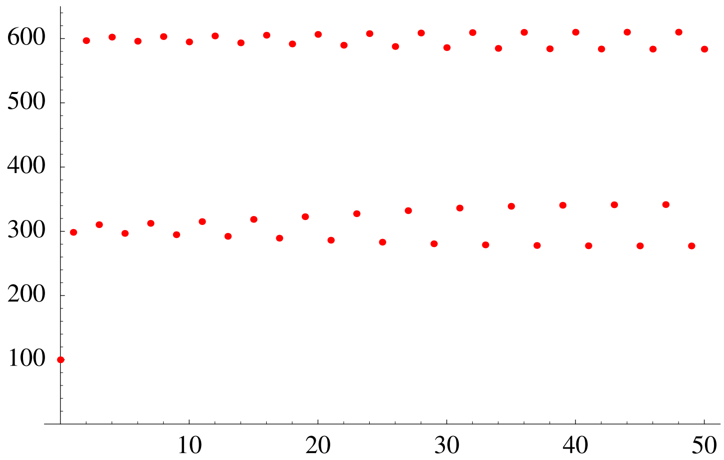

Figure 1.5.5 Inhibited population growth with $\beta = 2.48$

It is possible to obtain limiting cycles of longer periods by increasing β. For example, Figure 1.5.5 shows the effect of letting $\beta = 2.48$. Note that $\{x_n\}$ appears to be approaching a limiting cycle of period 4.

With appropriate choices for β and x_0, it is in fact possible for the inhibited growth model to exhibit limiting cycles of any given period. This is related to the fact that it is possible for this model to behave chaotically. Intuitively, a sequence is *chaotic* if it displays erratic behavior which, although in theory completely determined by a difference equation such as (1.5.6), is in practice unpredictable because small changes in the initial value x_0 yield strikingly different sequences. For example, Figures 1.5.6 and 1.5.7 illustrate the differing behavior of the inhibited growth model with $\beta = 2.95$, first for an initial population of 100 and then for an initial population of 101.

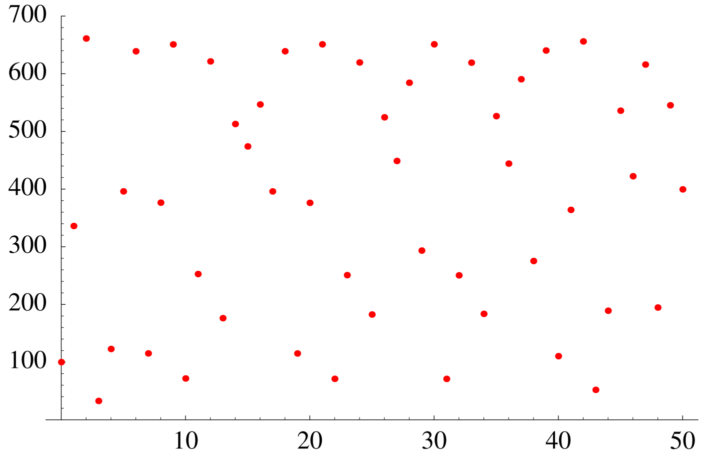

Figure 1.5.6 Inhibited population growth with $\beta = 2.95$ and $x_0 = 100$

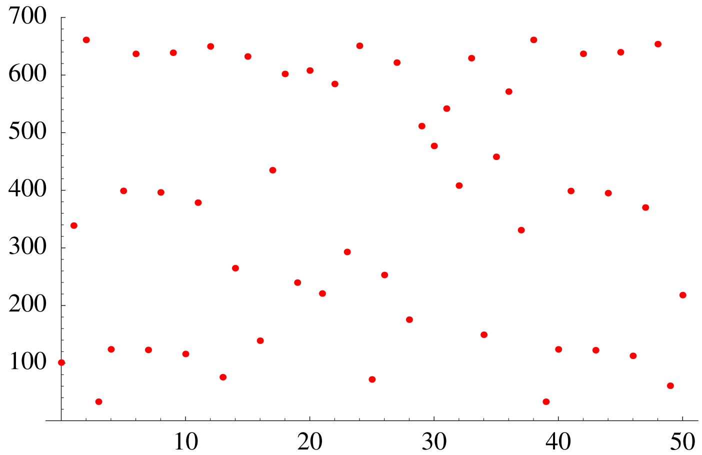

Figure 1.5.7 Inhibited population growth with $\beta = 2.95$ and $x_0 = 101$

Problems

1. A population of weasels has a natural growth rate of 3% per year. Let w_n be the number of weasels n years from now and suppose there are currently 300 weasels.
   (a) Suppose the carrying capacity of the weasel's habitat is 1000. Using an inhibited growth model, write a difference equation which describes how the population changes from year to year.
   (b) Using the difference equation from part (a), compute w_n for $n = 1, 2, \ldots, 150$.
   (c) How many years will it take for the population to double? To triple?
   (d) Plot w_n versus n for $n = 0, 1, 2, \ldots, 150$. From the plot, guess $\lim_{n \to \infty} w_n$.
   (e) Compare your answers with those to Problem 3 in Section 1.4.
2. Suppose a population of northern pike in a lake in Montana has a natural growth rate of 4.5% per year, but the lake can support no more than 10,000 pike. Let p_n be the

number of pike n years from now and suppose $p_0 = 1000$.

(a) Use the inhibited growth model to write a difference equation which describes how the population changes from year to year.

(b) Using the difference equation from part (a), compute p_n for $n = 1, 2, 3, \ldots 50$.

(c) How many years will it take for the population to double? To triple?

(d) Plot p_n versus n for $n = 0, 1, 2, \ldots 200$. From the plot, guess $\lim_{n\to\infty} p_n$.

(e) How many years will it take for the population to reach 9500?

3. Do Problem 2 assuming an uninhibited growth model and no restrictions on the number of pike that the lake can support.

4. Suppose r_n represents the number of snowshoe rabbits in a certain National Forest in Alaska after n years with an initial value of $r_0 = 5000$. Moreover, suppose the forest can support no more than 10,000 rabbits and $\{r_n\}$satisfies the inhibited growth model

$$r_{n+1} = r_n + \frac{\beta}{10,000} r_n(10,000 - r_n)$$

for $n = 0, 1, 2, \ldots$. For each of the following values for β, plot r_n versus n for $n = 0, 1, 2, \ldots, 100$ and comment on the behavior of the sequence, in particular noting any limiting values or limiting cycles

(a) $\beta = 0.5$
(b) $\beta = 1.5$
(c) $\beta = 2.4$
(d) $\beta = 2.5$
(e) $\beta = 2.56$
(f) $\beta = 2.9$

5. Using an initial value of $x_0 = 0.5$, let $\{x_n\}$ be the sequence which satisfies the difference equation

$$x_{n+1} = \mu x_n(1 - x_n),$$

$n = 0, 1, 2, \ldots$. Plot x_n versus n for the following values of μ and comment on the behavior of the sequence, in particular noting any limiting values or limiting cycles.

(a) $\mu = 0.9$
(b) $\mu = 1.0$
(c) $\mu = 1.5$
(d) $\mu = 2.0$
(e) $\mu = 2.5$
(f) $\mu = 3.0$
(g) $\mu = 3.1$
(h) $\mu = 3.5$
(i) $\mu = 3.57$
(j) $\mu = 1 + \sqrt{8}$
(k) $\mu = 3.99$
(l) $\mu = 4.0$

6. Repeat Problem 5 starting with an initial value of $x_0 = 0.6$.

7. If f is any function defined for real numbers, then the difference equation

$$x_{n+1} = f(x_n),$$

$n = 0, 1, 2, \ldots$, is called a *discrete dynamical system.* For any given initial condition x_0, the sequence $\{x_n\}$ which satisfies this equation is called an orbit of f. Note that an orbit of f is simply the sequence of points

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots.$$

For example, the difference equation in Problem 5 is an example of a discrete dynamical system with $f(x) = \mu x(1-x)$. For each of the following, compute 50 terms of the given orbit and discuss its behavior.

(a) $x_0 = 10,\ f(x) = 2x$

(b) $x_0 = 100,\ f(x) = 0.8x$

(c) $x_0 = 2\ f(x) = \cos(x)$

(d) $x_0 = 2,\ f(x) = \sin(x)$

(e) $x_0 = 5,\ f(x) = \dfrac{1}{2}\left(x + \dfrac{2}{x}\right)$

(f) $x_0 = 1,\ f(x) = \dfrac{2x^2}{3x^2 - 5}$

(g) $x_0 = 0,\ f(x) = x^2 + 1.0$

(h) $x_0 = 0,\ f(x) = x^2 - 0.5$

(i) $x_0 = 0,\ f(x) = x^2 - 0.8$

(j) $x_0 = 0,\ f(x) = x^2 - 1.0$

(k) $x_0 = 0,\ f(x) = x^2 - 1.9$

(l) $x_0 = 0,\ f(x) = x^2 - 2.0$

8. Assuming that the sequence $\{x_n\}$ satisfying the inhibited growth model equation

$$x_{n+1} = x_n + \frac{\beta}{M} x_n (M - x_n)$$

has a limit, show that $\lim\limits_{n \to \infty} x_n = M$.