COMMUTING PAIRS OF ISOMETRIES – GENERALIZED POWERS

ZBIGNIEW BURDAK, MAREK KOSIEK, PATRYK PAGACZ AND MAREK SŁOĈIŃSKI

1. Pairs defined by diagrams

The idea of pairs of isometries defined by a diagram appeared in [2]. Recall that a diagram is a set \( J \subset \mathbb{Z}^2 \) such that \((i, j) + J \subset J\) for any pair of nonnegative integers \((i, j)\) where \((i, j) + J := \{(x + i, y + j) : (x, y) \in J\}\).

**Definition 1.** Let \( J \) be a diagram and \( \mathcal{H} \) be a complex Hilbert space. Put

\[
H = \bigoplus_{(i, j) \in J} H_{i, j} \quad \text{where} \quad H_{i, j} = \mathcal{H}.
\]

For \( x = \{x_{i,j}\}_{(i,j) \in J} \in H \), define \( y = \{y_{i,j}\}_{(i,j) \in J} \in H \) and \( z = \{z_{i,j}\}_{(i,j) \in J} \in H \) by

\[
y_{i,j} = \begin{cases} 0, & (i-1, j) \notin J \\ x_{i-1,j}, & (i-1, j) \in J, \end{cases} \quad \text{and} \quad z_{i,j} = \begin{cases} 0, & (i, j-1) \notin J \\ x_{i,j-1}, & (i, j-1) \in J. \end{cases}
\]

Define isometries \( V_1 \) and \( V_2 \) on \( H \) by

\[
V_1 x = y \in H, \quad V_2 x = z \in H.
\]

We call them the isometries defined by diagram \( J \) and space \( \mathcal{H} \).

If \( \mathcal{H} \) in Definition 1 is one dimensional (\( \mathcal{H} = \mathbb{C} e_{i,j} \)), then the pair of isometries is called simple:

**Definition 2.** Let \( J \) be a diagram and \( H := \bigoplus_{(i, j) \in J} \mathbb{C} e_{i,j} \) where \( \{e_{i,j}\}_{(i,j) \in J} \) are orthonormal. A pair of isometries \( V_1, V_2 \in L(H) \) defined by \( V_1 e_{i,j} = e_{i+1,j} \), \( V_2 e_{i,j} = e_{i,j+1} \) for all \((i, j) \in J\) is called a simple pair of isometries given by diagram \( J \).

2. Generalized powers

A special type of pairs of isometries are generalized powers defined in [1]. We precede the formal definition by giving some properties and background of the idea of generalized powers. For every such pair it holds equality \( V_1^m = U V_2^m \) for some positive integers \( m, n \) and a unitary operator \( U \). The name “generalized powers” follows from a generalization of the example: \( V_1 = V^n, V_2 = V^m \) where \( V \) is a unilateral shift. In the example we have \( V_1^m = V_2^n \) and \( U = I \). An example of generalized powers which is the most different from the above one is when \( U \) is a bilateral shift. For respective wandering vectors such pair turns out to be a pair of generalized powers but also a pair defined by a diagram. We make a construction of such a pair starting from a diagram. Such a diagram has to be periodic in the following sense [1]:

**Definition 3.** The diagram \( J \) is periodic if there are positive numbers \( m, n \) such that for \( J_0 := \{(0, 1, \ldots, m - 1) \times \mathbb{Z}\} \cap J \) and \( J_k = J_0 + k(m, n) := \{(i + km, j - kn) : (i, j) \in J_0\} \) it holds \( J = \bigcup_{k \in \mathbb{Z}} J_k \), where \( J_k \) are disjoint for different \( k \). Then set \( J_0 \) is called a period of a diagram.

We now give a precise definition of generalized powers.

**Definition 4.** Let it be given:

1. a periodic diagram \( J = \bigcup_{k \in \mathbb{Z}} J_k \) with numbers \( m, n \),
(2) unitary operator $U \in L(H)$ and $e \in H$ such that $\bigvee\{U^n e : n \in \mathbb{Z}\} = H$.

Define:

1. Hilbert space $H := \bigoplus_{(i,j) \in J_0} H_{i,j}$ where $H_{i,j} = H$,
2. $\hat{U} \in L(H)$ where $(\hat{U} \oplus_{(i,j) \in J_0} x_{i,j}) = \oplus_{(i,j) \in J_0} U(x_{i,j})$,
3. $e_{i,j} \in H$ a vector such that $P_{H_{k,l}} e_{i,j} = e$ for $(k,l) = (i,j)$ and 0 otherwise, where $(i,j) \in J_0$,
4. $e_{i+km,j-kn} = \hat{U}^k e_{i,j}$ for $(i,j) \in J_0$ and $k \in \mathbb{Z}$,
5. $V_1(e_{i,j}) = e_{i+1,j}$, and $V_2(e_{i,j}) = e_{i,j+1}$.

Then operators $V_1, V_2$ are called a pair of generalized powers given by a unitary operator $\hat{U} \in L(H)$ and a diagram $J$.

**Proposition 5.** Generalized powers are pairs of unilateral shifts.

**References**


Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Prof. St. Lojasiewicza 6, 30-348 Kraków, Poland

E-mail address: Marek.Kosiek@im.uj.edu.pl