Ergodic properties of operators in some semi-Hilbertian spaces

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Nemecká, 6.09.2011

Cesàro ergodicity

An operator $T \in \mathcal{B}(\mathcal{H})$ is **Cesàro ergodic** if the sequence $\{M_n(T)\}_{n=1}^{\infty}$ of Cesàro averages

$$M_n(T) = \frac{1}{n}\sum_{j=0}^{n-1} T^j$$

of T is convergent in the strong operator topology, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j h = Ph, \quad h \in \mathcal{H},$$

where P is the oblique projection such that

$$\mathcal{N}(P) = \overline{\mathcal{R}(I-T)} \text{ and } \mathcal{R}(P) = \mathcal{N}(I-T).$$

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If T and T^* are Cesàro ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^{*j} h = P^* h, \quad h \in \mathcal{H}.$$

Orthogonally mean ergodic operator

If P is an orthogonal projection, then T is called an orthogonally mean ergodic operator.

Theorem

An operator $T \in \mathcal{B}(\mathcal{H})$ is Cesàro ergodic if and only if it is similar to an orthogonally mean ergodic operator on $\mathcal{H}$.

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Example

Let $C_{1,\cdot}(\mathcal{H})$ be the class of all power bounded operators $T \in \mathcal{B}(\mathcal{H})$ such that

$$\inf_{n\in\mathbb{N}} \|T^n h\| > 0, \quad h \in \mathcal{H} \setminus \{0\}.$$

It turns out that if $T \in C_{1,\cdot}(\mathcal{H})$, then T and T^* are orthogonally mean ergodic operators

(Tools from papers of L. Kérchy).

Classical results

$\|T\| \leq 1 \quad \Longrightarrow \quad T$ is orthogonally mean ergodic,

$\sup_{n\in\mathbb{N}} \|T^n\| < \infty \quad \Longrightarrow \quad T$ is Cesàro ergodic.

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We assume that $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, i.e.,

$$\langle Ah, h \rangle \geq 0, \quad h \in \mathcal{H}.$$

Such an A induces a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$\langle h, f \rangle_A = \langle Ah, f \rangle, \quad h, f \in \mathcal{H}.$$

Denote by $\| \cdot \|_A$ the seminorm induced by $\langle \cdot, \cdot \rangle_A$, i.e.,

$$\|h\|_A = \sqrt{\langle h, h \rangle_A}, \quad h \in \mathcal{H}.$$

We put

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \,|\, \exists c > 0 \, \forall h \in \mathcal{H} : \ \|Th\|_A \leq c\|h\|_A\}.$$

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We equip $\mathcal{B}_A(\mathcal{H})$ with the seminorm $\|\cdot\|_A$.

$$\|T\|_A = \sup_{h \in \overline{\mathcal{R}(A)},\, h \neq 0} \frac{\|Th\|_A}{\|h\|_A} = \sup_{\|h\|_A \leq 1} \|Th\|_A.$$

A-adjoint

For $T \in \mathcal{B}_A(\mathcal{H})$, an operator $S \in \mathcal{B}_A(\mathcal{H})$ is called an A-adjoint of T if

$$\langle Th, f \rangle_A = \langle h, Sf \rangle_A, \quad h, f \in \mathcal{H},$$

i.e., $AS = T^*A$.

We say that T is A-selfadjoint if $AT = T^*A$.

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Existence and uniqueness

If $T \in \mathcal{B}_A(\mathcal{H})$, then $S \in \mathcal{B}_A(\mathcal{H})$ such that

$$AS = T^*A$$

need not exist.

Note that $\mathcal{B}_{A^2}(\mathcal{H}) \subset \mathcal{B}_A(\mathcal{H})$

(S. Hassi, Z. Sebestyén, H.S.V. de Snoo, 2005).

If $T \in \mathcal{B}_{A^2}(\mathcal{H})$, then S exists, but it may not be uniquely determined.

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Theorem [R.G. Douglas, 1966]

For $L, R \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:

(i) $\mathcal{R}(R) \subset \mathcal{R}(L)$,

(ii) there exists a positive number λ such that $RR^* \leq \lambda LL^*$,

(iii) there exists $C \in \mathcal{B}(\mathcal{H})$ such that $LC = R$.

If (i) holds, then there exists a unique operator $S \in \mathcal{B}(\mathcal{H})$ such that

$$LS = R, \ \mathcal{R}(S) \subset \overline{\mathcal{R}(L^*)} \text{ and } \mathcal{N}(S) = \mathcal{N}(R).$$

If T has an A-adjoint S, then $AS = T^*A$.
By the Douglas theorem, we can find a unique T_A such that

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A-power boundedness

$T \in \mathcal{B}_A(\mathcal{H})$ is **A-power bounded** if $\sup_{n\in\mathbb{N}} \|T^n\|_A < \infty$.

Lemma

For $T \in \mathcal{B}_A(\mathcal{H})$ the following conditions are equivalent:

(a) T is A-power bounded,

(b) $T_{A^{1/2}}$ is power bounded on $\mathcal{H}$,

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Theorem

Let $T \in \mathcal{B}_A(\mathcal{H})$ be an A-power bounded operator. Then

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T_{A^{1/2}}^j h - Qh \right\|_A = 0, \quad h \in \mathcal{H}, \tag{1}$$

where $Q \in \mathcal{B}(\mathcal{H})$ is the ergodic projection of $T_{A^{1/2}}$. Moreover, $Q \in \mathcal{B}_A(\mathcal{H})$ if and only if there exists $P \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j h - Ph \right\|_A = 0, \quad h \in \mathcal{H}. \tag{2}$$

In this case, $P \in \mathcal{B}_A(\mathcal{H})$, P is an $A^{1/2}$-adjoint of Q, and $P_{A^{1/2}} = Q$.

In addition, if $T, P \in \mathcal{B}_{A^2}(\mathcal{H})$, then

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T_A^j h - P_A h \right\|_A = 0, \quad h \in \mathcal{H}, \tag{3}$$

and $(P_A)_{A^{1/2}} = Q^*$.

In what follows, we put

$$\mathcal{N} := \mathcal{N}(A^{1/2} - A^{1/2}T), \quad \mathcal{N}_* := \mathcal{N}(A^{1/2} - T^*A^{1/2}).$$

Theorem

Suppose that $T \in \mathcal{B}_A(\mathcal{H})$ is A-power bounded operator and the ergodic projection Q of $T_{A^{1/2}}$ belongs to $\mathcal{B}_A(\mathcal{H})$. Then TFAE:

(i) Q is $A^{1/2}$-selfadjoint, i.e., $A^{1/2}Q = Q^*A^{1/2}$,

(ii) $\lim_{n\to\infty} \|\frac{1}{n}\sum_{j=0}^{n-1} T^j h - Qh\|_A = 0$,

(iii) $\mathcal{N} = \mathcal{N}_*$,

(iv) $A^{1/2}\mathcal{N} = A^{1/2}\mathcal{N}_*$.

Definition

Under assumptions of the above theorem, if one of the above condition holds, then we say that T is A-ergodic.

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Suppose that $T \in \mathcal{B}_A(\mathcal{H})$ is A-power bounded operator and the ergodic projection Q of $T_{A^{1/2}}$ belongs to $\mathcal{B}_A(\mathcal{H})$.

Then TFAE:

(i) Q is $A^{1/2}$-selfadjoint, i.e., $A^{1/2}Q = Q^*A^{1/2}$,

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Definition

Under assumptions of the above theorem, if one of the above condition holds, then we say that T is A-ergodic.

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Suppose that $T \in \mathcal{B}_A(\mathcal{H})$ is an A-power bounded operator for some positive, injective operator A, such that $T_{A^{1/2}}$ is orthogonally mean ergodic. Then TFAE:

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Does always $(i) \Rightarrow (ii)$?

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When $A = I$, the previous theorem says that if an operator is A-ergodic, it is just orthogonally mean ergodic.

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Example

$\mathcal{H}$ - a separable Hilbert,
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$$Te_n := \frac{n+2}{n+1} e_{n+1}, \quad n \in \mathbb{N}.$$

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