Examples of non-hyperreflexive reflexive spaces of operators

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$$\mathrm{d}\,(T, \mathcal{S}) \le c\,\alpha\,(T, \mathcal{S}) \qquad \forall\, T \in \mathcal{L}(\mathcal{H}, \mathcal{H}'). \tag{1}$$

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In the following proposition (vi) is stated more precisely:

Proposition (Bessonov-Bračič-Zajac 2011)

Let $\mathcal{X}$ *and* $\mathcal{Y}$ *be complex Banach spaces and let* $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$ *be a hyperreflexive subspace of operators. If* $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *and* $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *are invertible operators,*
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Corollary

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ be a hyperreflexive linear space. If U and V are unitary operators on $\mathcal{H}$, then the space $U\mathcal{S}V$ is hyperreflexive and

$$\kappa(U\mathcal{S}V) = \kappa(\mathcal{S})\,. \tag{2}$$

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The converse (of 2.) was proved by K. Kliś and M. Ptak (2006):

Theorem

$\mathcal{S} = \bigoplus_{n=1}^{\infty} \mathcal{S}_n$ is hyperreflexive if and only if $\forall\, \mathcal{S}_n$ are hyperrefl. and $\exists\, K > 0$ s.t. $\kappa(\mathcal{S}_n) \leq K\ \forall\, n \in \mathbb{N}$.

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Let H_2 be a two-dimensional Hilbert space with orthonormal basis $\{e_1, e_2\}$. Fix $0 < \varepsilon < 1/3$ and put $u_1 = e_1$, $u_2 = e_1 + \varepsilon e_2$.

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$$\mathcal{S}_\varepsilon = \left\{ S_{\lambda,\mu} = \begin{pmatrix} 0 & \lambda \\ \mu & -(\lambda+\mu)/\varepsilon \end{pmatrix} : \ \lambda, \mu \in \mathbb{C} \right\} .$$

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(3) has been proved directly from the definition. Now, we can give more precise estimate.

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Let $\mathcal{H} = \mathbb{C}^2$ *and let* $\mathcal{L} \neq \mathcal{M}$ *be one-dimensional subspaces of* $\mathcal{H}$*, i.e.* $\mathcal{L} + \mathcal{M} = \mathcal{H}$*. Denote*

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$\mathcal{S}_\varepsilon$ from the Kraus-Larson example is not $\mathrm{Alg}\{\mathcal{L}, \mathcal{M}\}$ (from Tosaka). However it is unitary equivalent to such an algebra:

Proof of the lemma.

Observe that $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is unitary and for $\forall\, \lambda, \mu \in \mathbb{C}$

$$US_{\lambda,\mu} = U \begin{pmatrix} 0 & \lambda \\ \mu & -(\lambda+\mu)/\varepsilon \end{pmatrix} = \begin{pmatrix} -\mu & (\lambda+\mu)/\varepsilon \\ 0 & \lambda \end{pmatrix} .$$

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where $\cos\varphi = \frac{(u_1, u_2)}{\|u_1\|\|u_2\|} = \frac{1}{\sqrt{1+\varepsilon^2}}$.

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$$US_{\lambda,\mu} = U \begin{pmatrix} 0 & \lambda \\ \mu & -(\lambda+\mu)/\varepsilon \end{pmatrix} = \begin{pmatrix} -\mu & (\lambda+\mu)/\varepsilon \\ 0 & \lambda \end{pmatrix} .$$

$$\text{Putting } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

we obtain $U\mathcal{S}_\varepsilon = \mathrm{Alg}\{[u_1], [u_2]\}$, $u_1 = e_1, u_2 = e_1 + \varepsilon e_2$, and

$$\kappa(\mathcal{S}_\varepsilon) = \frac{1}{\sin\varphi} ,$$

where $\cos\varphi = \frac{(u_1, u_2)}{\|u_1\|\|u_2\|} = \frac{1}{\sqrt{1+\varepsilon^2}}$.

$\sin\varphi = \sqrt{1-\cos^2\varphi} = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \implies \frac{1}{\sin\varphi} = \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} > \frac{1}{\varepsilon}$ .. □

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we obtain $X \in I(A_n, B_n) \iff \exists\, \lambda, \mu \in \mathbb{C} : X_n = \left(\begin{smallmatrix} 0 & \lambda \\ \mu & -n(\lambda+\mu) \end{smallmatrix}\right)$,

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Now it is easy to prove

Theorem (M.Z. 2008)

There exist operators T, T' for which $I(T, T')$ is reflexive but not hyperreflexive.

Proof.

It is enough to put

$$T_n = e^{i\pi/n}\frac{1}{n}(nI + A_n)\,, \qquad T_n' = e^{i\pi/n}\frac{1}{n}(nI + B_n)\,.$$

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- For $n \neq m$ the minimal polynomials of T_n and T_m' are relatively prime,
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Thus the Kraus-Larson example is also an example of intertwiner which is reflexive but not hyperreflexive. □

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(i) $\|T_n\| = 1$.

(ii) $D_n\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1/n \\ 1-(1/n)+(1/n^2) \end{smallmatrix}\right) \implies \|D_n\| > 1 - (1/n) + (1/n^2)$,

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(iv) If $m \neq n$ then $\sigma(T_n) \cap \sigma(T_m) = \emptyset$.

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6. *Blaschke product* $B(\lambda) = \prod\limits_{k=1}^{\infty} \frac{\overline{\lambda_k}}{|\lambda_k|} \frac{\lambda_k - \lambda}{1 - \overline{\lambda_k}\lambda} \frac{\overline{\mu_k}}{|\mu_k|} \frac{\mu_k - \lambda}{1 - \overline{\mu_k}\lambda}$ *converges in the open unit disk.*

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Consequently, $T = \bigoplus_{k=1}^{\infty} T_k$ *is a* C_0 *contraction having minimal function* $B(\lambda)$ *and* $\{T\}'$ *is reflexive but not hyperreflexive.*

Recall that $m(\lambda) \in H^\infty$ is the minimal function of a C_0 contraction T if $m(T) = 0$ and if $f(T) = 0$, then $m|f$. The simplest C_0 is model operator S_m:

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Now, a construction of reflexive but not hyperreflexive S_m will be indicated:

First, we recall a sufficient condition for hyperreflexivity of the model operator

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Then the angle that make those eigenvalues $\to 0$ and, consequently

$$\lim \kappa(S_B|M_n) = \infty\,.$$

So this example is again of the Kraus-Larson type.

We conclude with a natural open problem:

Question

Does there exist a non-hyperreflexive reflexive space of operators which is not similar to a direct sum of reflexive spaces?

Thank you for your attention

Thank you for participating in 8th WFA