On multiple solutions of generalized second order boundary value problem with Φ–Laplacian.

Boris Rudolf

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Introduction

Differential equation with Φ–Laplacian

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  $x(t) \in D = \{x \in C^1(I),\ \Phi(x'(t)) \in C^1(I)\}$,
- Multiple solutions.

Lower and upper solution.

Lower solution $\alpha \in D^0$,

$$\lim_{t \to t_i -} \alpha'(t) \le \lim_{t \to t_i +} \alpha'(t) \qquad \text{for i=1,...,n,}$$

$$(\Phi(\alpha'(t)))' \ge f(t, \alpha(t), \Phi(\alpha'(t))) \qquad \text{for } t \in I^0,$$

$$\alpha'(0) \ge 0, \qquad \alpha(b) \le \int_0^b \alpha(s) dg(s) - k\Phi(\alpha'(b)).$$

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Upper solution $\beta \in D^0$,

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$$(\Phi(\beta'(t)))' \leq f(t, \beta(t), \Phi(\beta')), \qquad \text{for } t \in I^0,$$

$$\beta'(0) \leq 0, \qquad \beta(b) \geq \int_0^b \beta(s)dg(s) - k\Phi(\beta'(b)).$$

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Let α, β be a strict lower and upper solution and $x(t)$ be a solution of the boundary value problem (1), (2).

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Let $\forall r > r_0$, $\exists a_r > 0$ and a function $h_r \in C(R_0^+, [a_r, \infty])$ satisfying

$$\int_0^\infty \frac{\Phi^{-1}(s)}{h_r(s)}\, ds = \infty, \quad \int_{-\infty}^0 \frac{\Phi^{-1}(s)}{h_r(|s|)}\, ds = -\infty \tag{3}$$

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for $t \in I$, $|x| < r$, $y \in R$.
Then $\forall r > r_0$, $\exists \rho_r > 0$ such that for a solution x of (1), (2) $||x|| < r$ implies $||x'|| < \rho_r$.

Existence results.

Theorem.

Let $r > 0$ be such that

- $f(t, r, 0) > 0$ *and* $f(t, -r, 0) < 0$ *on* I,

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$$Tx(t) = \frac{1}{G(b) - 1} \left\{ \int_0^b G(s)\Phi^{-1}(F_x(s))\, ds + k(F_x(b)) \right\} - \int_t^b \Phi^{-1}(F_x(s))\, ds.$$

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$$F_x(s) = \int_0^s f(\tau, x(\tau), \Phi(x'(\tau)))\, d\tau$$

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Then there exists a solution x *of (1),(2) such that* $\exists t_a \in I,\ \alpha(t_a) > x(t_a),\ \exists t_b \in I,\ x(t_b) > \beta(t_b)\}$.

Multiplicity results.

Lemma.

Let α be a strict lower solution of the problem (1), (2).
Set

$$f_\alpha(t,x,y) = \begin{cases} f(t,x,y) & x(t) > \alpha(t) \\ f(t,\alpha(t),y) & x(t) \le \alpha(t). \end{cases}$$

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Examples

Example.

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\begin{aligned}
(\varphi_p(x'))' + f(t,x) &= 0, \\
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- $\exists$ *constants* x_1, x_2, $x_1 < x_2$, *such that* $f_1(t,x_1) < f_1(t,x_2)$ *for each* $t \in I$.

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- $f_1(t, x_1) \leq h(t) \leq f_1(t, x_2) \Rightarrow$ at least two solutions,
- for each $h(t) \in C(I) \Rightarrow$ exists a solution.

Thank you for your attention.