One-dimensional and multidimensional spectral order

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1. Spectral order $\preccurlyeq$
   - Introduction
   - $\preccurlyeq$ and $\leqslant$-comparison
2. Multidimensional spectral order
   - General case
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- If the operator A is closable, we denote by $\bar{A}$ its closure.

Definitions

- Denote by $\boldsymbol{B}(\mathcal{H})$ the C^*-algebra of all bounded operators A in $\mathcal{H}$ with $\mathscr{D}(A) = \mathcal{H}$. As usual, $I = I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$.

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- $\boldsymbol{B}_s(\mathcal{H}) = \{A \in \boldsymbol{B}(\mathcal{H})\colon A = A^*\}$
- Given two selfadjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$, we write $A \leqslant B$ whenever $\langle Ah, h\rangle \leqslant \langle Bh, h\rangle$ for all $h \in \mathcal{H}$.

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- A densely defined operator A in $\mathcal{H}$ is said to be *selfadjoint* if $A = A^*$ and *positive* if $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathscr{D}(A)$.

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- The last definition of $\leqslant$ is easily seen to be consistent with that for bounded operators.

Remark

In general inequality $0 \leqslant A \leqslant B$, where $A, B \in \boldsymbol{B}(\mathcal{H})$, may not imply $A^n \leqslant B^n$, where $n \in \mathbb{N}$.

Theorem (M.P. Olson, A. P., J. Stochel)

Let A and B be positive selfadjoint operators in $\mathcal{H}$. Then the following conditions are equivalent:

(i) $A^n \leqslant B^n$ *for all* $n \in \mathbb{N}$,

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(iii) $A \preccurlyeq B$.

Let us consider two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$.Let A and B_θ be the matrices given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B_\theta = \begin{bmatrix} 2 & 1 \\ 1 & \theta \end{bmatrix} \text{ for } \theta \in [1, \infty). \tag{1}$$

Clearly, $A \geqslant 0$ and $B_\theta \geqslant 0$.

Proposition

Let A and B_θ be as in (1). Then for every positive integer k there exists $\theta_k \in (2, \infty)$ such that for all $\theta \in [\theta_k, \infty)$,

(i) $A^n \leqslant B_\theta^n$ *for all* $n = 0, \ldots, k$,

(ii) $A \not\preccurlyeq B_\theta$.

The definition of spectral order

- Let $A, B \in \boldsymbol{B}_s(\mathcal{H})$ with spectral measure E_A and E_B, respectively. We write $A \preccurlyeq B$ if $E_B((-\infty, x]) \leqslant E_A((-\infty, x])$ for all $x \in \mathbb{R}$.

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- The relation $\preccurlyeq$ is a partial order in the set of all selfadjoint operators in $\mathcal{H}$.
- This definition was introduced in 1971 by Olson.

Lattices

- Kadison (1951): $(\boldsymbol{B}_s(\mathcal{H}), \leqslant)$ is an anti-lattice, i.e., for any $A, B \in \boldsymbol{B}_s(\mathcal{H})$, the supremum of the set $\{A, B\}$ exists if and only if A, B are comparable (either $A \leqslant B$ or $B \leqslant A$).

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- Sherman (1951): If the set of all selfadjoint elements of a C^*-algebra $\mathcal{A}$ with the usual order forms a lattice, then $\mathcal{A}$ is commutative.

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- Sherman (1951): If the set of all selfadjoint elements of a C^*-algebra $\mathcal{A}$ with the usual order forms a lattice, then $\mathcal{A}$ is commutative.
- Olson (1971): If $\mathcal{S}$ is the set of all selfadjoint elements of a von Neumann algebra $\mathscr{V}$ in $\boldsymbol{B}(\mathcal{H})$ then, $(\mathcal{S}, \preccurlyeq)$ is a conditionally complete lattice.

The definition of spectral order for unbounded operators

Given two selfadjoint operators A and B in $\mathcal{H}$ with spectral measure E_A and E_B, respectively, we write $A \preccurlyeq B$ if $E_B((-\infty, x]) \leqslant E_A((-\infty, x])$ for all $x \in \mathbb{R}$.

In the case of unbounded operators closed supports of E_A and E_B are not compact.

Proposition

Let A and B be selfadjoint operators in $\mathcal{H}$ such that $A \preccurlyeq B$. Then $\langle Ah, h\rangle \leqslant \langle Bh, h\rangle$ for all $h \in \mathscr{D}(A) \cap \mathscr{D}(B)$. Moreover, if A and B are bounded from below, then $\mathscr{D}(B) \subseteq \mathscr{D}(A)$.

Remark

In general, the relation $A \preccurlyeq B$ implies neither $\mathscr{D}(B) \subseteq \mathscr{D}(A)$ nor $\mathscr{D}(A) \subseteq \mathscr{D}(B)$. It is even possible to find operators A and B such that $A \preccurlyeq B$ and $\mathscr{D}(A) \cap \mathscr{D}(B) = \{0\} \neq \mathcal{H}$.

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Theorem (M. P. Olson, M. Fujii, I. Kasahara, A. P.,J. Stochel)

If A and B are selfadjoint operators in $\mathcal{H}$, then the following conditions are equivalent:

(i) $A \preccurlyeq B$,

(ii) $f(A) \leqslant f(B)$ *for each bounded continuous monotonically increasing function* $f : \mathbb{R} \to [0, \infty)$,

(iii) $f(A) \leqslant f(B)$ *for each bounded monotonically increasing function* $f : \mathbb{R} \to \mathbb{R}$.

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- An element of

$$\mathscr{B}(A) = \bigcup_{a>0} \{h \in \mathscr{D}^\infty(A) \colon \exists_{c>0} \forall_{n\geqslant 0} \|A^n h\| \leqslant c a^n\}$$

is called a *bounded vector* of A.

Theorem

If A and B are positive selfadjoint operators in $\mathcal{H}$, then the following conditions are equivalent:

(i) $A \preccurlyeq B$,

(ii) $\mathscr{D}^\infty(B) \subseteq \mathscr{D}^\infty(A)$ *and the set* $\mathscr{I}_{A,B}(h)$ *is unbounded for all* $h \in \mathscr{D}^\infty(B)$,

(iii) $\mathscr{B}(B) \subseteq \mathscr{D}^\infty(A)$ *and the set* $\mathscr{I}_{A,B}(h)$ *is unbounded for all* $h \in \mathscr{B}(B)$,

(iv) $\mathscr{B}(B) \subseteq \mathscr{B}(A)$ *and the set* $\mathscr{I}_{A,B}(h)$ *is unbounded for all* $h \in \mathscr{B}(B)$,

where $\mathscr{I}_{A,B}(h) := \{s \in [0,\infty) \colon \langle A^s h, h\rangle \leqslant \langle B^s h, h\rangle\}$ *for* $h \in \mathscr{D}^\infty(A) \cap \mathscr{D}^\infty(B)$.

Recall that due to Stone's theorem the infinitesimal generator of a C_0-semigroup of bounded selfadjoint operators on $\mathcal{H}$ is always selfadjoint.

Theorem

Let $\{T_j(t)\}_{t \geqslant 0} \subseteq \boldsymbol{B}(\mathcal{H})$ be a C_0-semigroup of selfadjoint operators and A_j be its infinitesimal generator, $j = 1, 2$. Then the following conditions are equivalent:

(i) $A_1 \preccurlyeq A_2$,

(ii) $T_1(t) \preccurlyeq T_2(t)$ *for some* $t > 0$,

(iii) $T_1(t) \preccurlyeq T_2(t)$ *for every* $t > 0$,

(iv) $T_1(t) \leqslant T_2(t)$ *for some* $t > 0$ *and* $E_A((-\infty, x])E_B((-\infty, x]) = E_B((-\infty, x])E_A((-\infty, x])$ *for every* $x \in \mathbb{R}$,

(v) $T_1(nt) \leqslant T_2(nt)$ *for some* $t > 0$ *and for infinitely many* $n \in \mathbb{N}$.

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- In the multidimensional case we restrict ours considerations to κ-tuples of selfadjoint operators, which consists of commuting operators.
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- $E_{\mathbf{A}}$-joint spectral measure of $\mathbf{A} = (A_1, \ldots, A_\kappa)$,

Definition

Let $\mathbf{A} = (A_1, \ldots, A_\kappa)$ and $\mathbf{B} = (B_1, \ldots, B_\kappa)$ be a κ-tuples of commuting selfadjoint operators in $\mathcal{H}$. We write $\mathbf{A} \preccurlyeq \mathbf{B}$ if $E_{\mathbf{B}}((-\infty, x]) \leqslant E_{\mathbf{A}}((-\infty, x])$ for every $x = (x_1, \ldots, x_\kappa) \in \mathbb{R}^\kappa$, where $(-\infty, x] := (-\infty, x_1] \times \ldots \times (-\infty, x_\kappa]$.

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- $|\alpha| := \alpha_1 + \ldots + \alpha_\kappa$ for $\alpha = (\alpha_1, \ldots, \alpha_\kappa) \in [0, \infty)^\kappa$,
- $x^\alpha := x_1^{\alpha_1} \ldots x_\kappa^{\alpha_\kappa}$ for $x = (x_1, \ldots, x_\kappa)$ and $\alpha = (\alpha_1, \ldots, \alpha_\kappa)$.

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be* κ*-tuples of commuting selfadjoint operators in* $\mathcal{H}$ *such, that* $\mathbf{A} \preccurlyeq \mathbf{B}$*. If* $\varphi \in S(\mathbb{R}^\kappa, E_{\boldsymbol{A}}) \cap S(\mathbb{R}^\kappa, E_{\boldsymbol{B}})$ *is separately monotonically increasing Borel function, then* $\varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B})$*. In particular* $\varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B})$ *for every separately monotonically increasing Borel function* $\varphi \colon \mathbb{R}^\kappa \to \mathbb{R}$*.*

Remark

Suppose that $\dim \mathcal{H} \geqslant 1$. Then each Borel function $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$ satisfying condition

$$\mathbf{A} \preccurlyeq \mathbf{B} \implies \varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B}) \tag{2}$$

for every $\mathbf{A}$, $\mathbf{B}$ κ-tuples of commuting selfadjoint operators, has to be separately monotonically increasing.

Corollary

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* κ*-tuples of commuting selfadjoint operators. Then the following conditions are equivalent:*

(i) $\mathbf{A} \preccurlyeq \mathbf{B}$,

(ii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded continuous function* $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$,

(iii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded Borel function* $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$.

Remark

Olson proved that the spectral order is not a vector order. In particular the implication $A \preccurlyeq B \implies A + C \preccurlyeq B + C$ does not hold for some $A, B, C \in \boldsymbol{B}_s(\mathcal{H})$. However spectral order has still some traces of vector order properties.

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Corollary

Let (A_1, A_2) and (B_1, B_2) be pairs of commuting selfadjoint operators in $\mathcal{H}$. Assume that $A_1 \preccurlyeq B_1$ and $A_2 \preccurlyeq B_2$. Then

$$\overline{A_1 + A_2} \preccurlyeq \overline{B_1 + B_2}.$$

Let

$$X^{\alpha}(\mathbf{A}) = \int_{\mathbb{R}^{\kappa}} x^{\alpha} dE_{\mathbf{A}}(x) = \overline{A_1^{\alpha_1} \dots A_{\kappa}^{\alpha_{\kappa}}},$$

for $\alpha \in \mathbb{N}^{\kappa}$.

What are the connections between the domains of operators $X^{\alpha}(\mathbf{A})$ and $X^{\alpha}(\mathbf{B})$, if $\mathbf{A} \preccurlyeq \mathbf{B}$?

Let

$$\mathbf{C}_\epsilon := (C_1^{\epsilon_1}, \ldots, C_\kappa^{\epsilon_\kappa}),$$

for $\mathbf{C} = (C_1, \ldots, C_\kappa)$ - κ-tuples of commuting selfadjoint operators in $\mathcal{H}$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\kappa) \in \{-,+\}^\kappa$, where $C^\pm := \int_{\mathbb{R}} x^\pm dE_C(x)$.

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ and $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ be a κ-tuples of commuting selfadjoint operators such that $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and $\alpha \in \mathbb{N}^\kappa$. If

$$X^\alpha(\boldsymbol{A}_\epsilon) \in \boldsymbol{B}(\mathcal{H}), \quad \epsilon \in \{-,+\}^\kappa \setminus \{(+, \ldots, +)\},$$

then

$$\mathscr{D}(X^\alpha(\boldsymbol{B})) \subset \mathscr{D}(X^\alpha(\boldsymbol{A})). \tag{3}$$

Condition

$$X^{\alpha}(\mathbf{A}_{\epsilon}) \in \boldsymbol{B}(\mathcal{H}), \quad \epsilon \in \{-,+\}^{\kappa} \backslash \{(+,\dots,+)\},$$

can't be weakened.

Example

For every $\epsilon \neq (+,\dots,+)$ we can find **A** and **B** such that $\mathbf{A} \preccurlyeq \mathbf{B}$ and

1. $X^{\alpha}(\mathbf{A}_{\delta}) \in \boldsymbol{B}(\mathcal{H})$ for every $\delta \in \{-,+\}^{\kappa} \backslash \{\epsilon\}$ and $\alpha \in \mathbb{N}_*^{\kappa}$,
2. $\mathscr{D}(X^{\alpha}(\mathbf{B})) \not\subset \mathscr{D}(X^{\alpha}(\mathbf{A}))$ for every $\alpha \in \mathbb{N}_*^{\kappa}$.

Let $\mathbf{A} = (A_1, \ldots, A_\kappa)$ and $\mathbf{B} = (B_1, \ldots, B_\kappa)$ be κ- tuples of commuting positive selfadjoint operators in $\mathcal{H}$. Define the set

$$\Lambda(\mathbf{A}, \mathbf{B}) := \{\alpha \in [0, \infty)^\kappa \colon X^\alpha(\mathbf{A}) \leqslant X^\alpha(\mathbf{B})\}.$$

We know that relation $\mathbf{A} \preccurlyeq \mathbf{B}$ implies the equality $\Lambda(\mathbf{A}, \mathbf{B}) = [0, \infty)^\kappa$.
What should be assumed about $\Lambda(\mathbf{A}, \mathbf{B})$ to have the reverse implication?

Without any additional informations about **A** and **B** we can formulate the following

Proposition

If $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *are* κ*-tuples of commuting positive selfadjoint operators in* $\mathcal{H}$*, then the following conditions are equivalent*

(i) $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$,

(ii) *for every* $j = 1, \ldots, \kappa$ *the set* $\Lambda(\boldsymbol{A}, \boldsymbol{B}) \cap \{se_j \colon s \in [0, \infty)\}$*, where* $e_j = (0 \ldots, \underbrace{1}_{j}, \ldots, 0)$*, is unbounded.*

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be* κ*-tuples of commuting positive selfadjoint operators such that* $\mathscr{N}(A_j) = \{0\}$ *for* $j = 1, \ldots, \kappa$. *Then the following conditions are equivalent:*

(i) $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$,

(ii) $\Lambda(\boldsymbol{A}, \boldsymbol{B}) = [0, \infty)^\kappa$,

(iii) $\sup_{\alpha \in \Lambda(\boldsymbol{A}, \boldsymbol{B})} \frac{\alpha_j}{1 + |\alpha| - \alpha_j} = \infty, \quad j = 1, \ldots, \kappa.$