On a dynamic contact problem for a geometrically nonlinear viscoelastic shell

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Abstract: We deal with an initial-boundary value problem describing the perpendicular vibrations of a viscoelastic Kármán-Donnell shell with a rigid inner obstacle. A weak formulation of a problem is in a form of a hyperbolic variational inequality. We solve this problem using the penalization method

1 INTRODUCTION AND NOTATION

Contact problems represent an important but complex topic of applied mathematics. For elastic problems there is only a very limited amount of results available (cf. [?] [?] and there cited literature). Viscosity makes possible to prove the existence of solutions for a broader set of problems for membranes, bodies as well as for linear models of plates. The von Kármán plate made of a short memory material in a dynamic contact was studied in [?]. The aim of the present paper is to extend these results to the nonlinear von Kármán-Donnell shells. The presented results also extend the research made for the quasistatic contact problems for viscoelastic shells (cf. [?]).

The existence of solutions is proved for an approximate penalized problem at first. The limit process to the original problem is enabled by an L_1 estimate of the penalty term and by the use of the compact imbedding theorem and by a proper use of the interpolation technique.

Let $\Omega \subset R^2$ be a bounded convex polygonal or C^2 domain with a boundary Γ and $I \equiv (0,T)$ be a bounded time interval. The unit outer normal vector is denoted by $\mathbf{n} = (n_1, n_2)$, $\tau = (-n_2, n_1)$ is the unit tangent vector. The displacement is denoted by $\mathbf{u} \equiv (u_i)$. Further employed notations are $\frac{\partial}{\partial s} \equiv \partial_s,\ \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr},\ \partial_i = \partial_{x_i},\ i = 1, 2, 3,$
$\dot{v} = \frac{\partial v}{\partial t},\ \ddot{v} = \frac{\partial^2 v}{\partial t^2},\ Q = I \times \Omega,\ S = I \times \Gamma.$

A shallow isotropic shell is occupying the domain

$$\mathcal{A} = \{(x, z) \in R^3 : x = (x_1, x_2) \in \Omega, |z - \mathcal{S}(x)| < h\},$$

where $z = \mathcal{S}(x),\ x \in \Omega$ is a middle surface of a shell.

Strain tensor is defined as

$$\begin{aligned} \varepsilon_{ij}(\mathbf{u}) &= \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - x_3 \partial_{ij} u_3,\ i, j = 1, 2; \\ \varepsilon_{i3} &\equiv 0,\ i = 1, 2, 3 \end{aligned}$$

with $k_{12} = 0$ and the curvatures $k_{ii} > 0,\ i = 1,\ 2.$

Further, we denote

$$[u, v] \equiv \partial_{11} u \partial_{22} v + \partial_{22} u \partial_{11} v - 2\partial_{12} u \partial_{12} v.$$

In the sequel, we denote by $W_p^k(M),\ k \geq 0,\ p \in [1, \infty]$ the Sobolev (for a noninteger p the Sobolev-Slobodetskii) spaces defined on a domain or an appropriate manifold M. By $\mathring{W}_p^k(M)$

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the spaces with zero traces are denoted. If $p = 2$ we use the notation $H^k(M)$, $\overset{\circ}{H}{}^k(M)$. The duals to $\overset{\circ}{H}{}^k(M)$ are denoted by $H^{-k}(M)$. For the anisotropic spaces $W_p^k(M)$, $k = (k_1, k_2) \in R_+^2$, $R_+ = (0, \infty)$, k_1 is related with the time while k_2 with the space variables (with the obvious consequences for $p = 2$). By C we denote the space of continuous functions with the appropriate sup-norm. By $\mathcal{H}$, $\overset{\circ}{\mathcal{H}}$ we denote the spaces $L_\infty(I; H^2(\Omega))$, $L_\infty(I; \overset{\circ}{H}{}^2(\Omega))$, respectively. The following generalization of the Aubin's compactness lemma verified in [**?**] Theorem 3.1 will be essentially used:

Lemma 1.1 *Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ be Banach spaces, the first reflexive and separable. Let $1 < p < \infty$, $1 \le q < \infty$. Then*

$$W \equiv \{v;\ v \in L_p(I; B_0),\ \dot{v} \in L_q(I, B_1)\} \hookrightarrow\hookrightarrow L_p(I; B).$$

2 CONTACT OF A VISCOELASTIC SHELL WITH A SHORT MEMORY

The constitutional law has the form

$$\begin{aligned} \sigma_{ij}(\mathbf{u}) = \ & \tfrac{E_1}{1-\mu^2}\partial_t\big((1-\mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{\mathbf{ij}}\varepsilon_{\mathbf{kk}}(\mathbf{u})\big) \\ & + \tfrac{E_0}{1-\mu^2}\big((1-\mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{\mathbf{ij}}\varepsilon_{\mathbf{kk}}(\mathbf{u})\big). \end{aligned}$$

The constants E_0, $E_1 > 0$ are the Young modulus of elasticity and the modulus of viscosity, respectively, $\mu \in (0, \frac{1}{2})$ is the Poisson ratio. We involve also the rotation inertia expressed by the term $a\Delta\ddot{u}$ in the first equation of the considered system with $a = \frac{h^2}{12}$. It will play the crucial role in the deriving a strong convergence of the sequence of velocities $\{\dot{u}_m\}$ in the appropriate space. Further we denote $b = \frac{h^2}{12\rho(1-\mu^2)}$ the material constant with $\rho > 0$ the density of the material. We concentrate for simplicity on the case of a free plate.

The classical formulation generalizes the elastic case derived in [**?**] and is composed of the system

$$\left.\begin{aligned} & \ddot{u} + a\Delta\ddot{u} + b(E_1\Delta^2\dot{u} + E_0\Delta^2 u) - [u, v] - k_{11}\partial_{22}v - k_{22}\partial_{11}v = f + g, \\ & u \ge 0,\ g \ge 0,\ ug = 0, \\ & \Delta^2 v + \\ & E_1\partial_t(\tfrac{1}{2}[u, u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) + E_0(\tfrac{1}{2}[u, u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) = 0 \end{aligned}\right\} \text{ on } Q, \tag{1}$$

the boundary conditions

$$\begin{aligned} & u \ge 0,\ \Sigma_1(u) \ge 0,\ u\Sigma_1(u) = 0, \\ & \mathcal{M}_1(u) = 0,\ v = 0 \text{ and } \partial_n v = 0 \text{ on } S, \\ & \mathcal{M}_1(u) = b[E_1 M(\dot{u}) + E_0(u)], \\ & \Sigma_1(u) = b[E_1 V(\dot{u}) + E_0 V(u)] - a\ddot{u} \end{aligned} \tag{2}$$

and the initial conditions

$$u(0, \cdot) = u_0 \ge 0,\ \dot{u}(0, \cdot) = u_1 \text{ on } \Omega. \tag{3}$$

We introduce cone $\mathcal{K} := \{y \in H^{1,2}(Q);\ \dot{y} \in L_2(I; H^1(\Omega)),\ y \ge 0\}$ and bilinear form

$$A(u, y) = \partial_{11}u\partial_{11}y + \partial_{22}u\partial_{22}y + \mu\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y + 2(1-\mu)\partial_{12}u\partial_{12}y\,.$$

Then the variational formulation of the problem (**??**–**??**) has the following form:

Find $\{u,v\} \in \mathcal{K} \times L_2(I; \mathring{H}^2(\Omega))$ *such that* $\dot{u} \in L_2(I; H^2(\Omega))$ *and the following system*

$$\begin{aligned}
&\int_Q (E_1 A(\dot{u}, y_1 - u) + E_0 A(u, y_1 - u) - ([u,v] + k_{11}\partial_{22}v + k_{22}\partial_{11}v)(y_1 - u))\, dx\, dt \\
&- \int_Q (a\nabla\dot{u} \cdot \nabla(\dot{y}_1 - \dot{u}) + \dot{u}(\dot{y}_1 - \dot{u}))\, dx\, dt \\
&+ \int_\Omega (a\nabla\dot{u} \cdot \nabla(y_1 - u) + \dot{u}(y_1 - u))\,(T,\cdot)\, dx \qquad (4)\\
\geq\ & \int_\Omega (a\nabla u_1 \cdot \nabla(y_1(0,\cdot) - u_0) + u_1(y_1(0,\cdot) - u_0))\, dx + \int_Q f(y_1 - u)\, dx\, dt, \\
&\int_\Omega \Delta v \Delta y_2\, dx = \qquad (5)\\
&- \int_\Omega \Big(E_1 \partial_t (\frac{1}{2}[u,u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) + E_0(\frac{1}{2}[u,u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) \Big) y_2\, dx
\end{aligned}$$

is satisfied for all $(y_1, y_2) \in \mathcal{K} \times \mathring{H}^2(\Omega)$.

We define the bilinear operator $\Phi : H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$ and the linear operators $\Delta_k : H^2(\Omega) \mapsto L_2(\Omega),\ L : H^2(\Omega) \to \mathring{H}^2(\Omega)$ by means of the variational equations and the identity

$$\int_\Omega \Delta\Phi(u,v)\Delta\varphi\, dx = \int_\Omega [u,v]\varphi\, dx \ \forall\varphi \in \mathring{H}^2(\Omega), \qquad (6)$$

$$\Delta_k v = k_{11}\partial_{22}v + k_{22}\partial_{11}v \ \forall v \in H^2(\Omega), \qquad (7)$$

$$\int_\Omega \Delta L u \Delta\varphi\, dx = \int_\Omega \Delta_k u \varphi\, dx \ \forall\varphi \in \mathring{H}^2(\Omega). \qquad (8)$$

The equation (**??**) has a unique solution, because $[u,v] \in L_1(\Omega) \hookrightarrow H^2(\Omega)^*$. The well-defined operator Φ is evidently compact and symmetric. The domain Ω fulfils the assumptions enabling us to apply Lemma 1 from [**?**] due to which $\Phi : H^2(\Omega)^2 \to W_p^2(\Omega),\ 2 < p < \infty$ and

$$\|\Phi(u,v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)} \ \forall u \in H^2(\Omega),\ v \in W_p^1(\Omega). \qquad (9)$$

The right-hand side of the equation (**??**) represents the linear bounded functional over $\mathring{H}^2(\Omega)$ and hence the operator $L : H^2(\Omega) \mapsto \mathring{H}^2(\Omega)$ is uniquely defined. Moreover it is compact due to the compact imbedding $H^1(\Omega) \hookrightarrow\hookrightarrow H^2(\Omega)$. Further it fulfils $L : H^2(\Omega) \mapsto W_p^2(\Omega),\ \ 2 < p < \infty$ and

$$\|Lu\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)} \ \forall u \in H^2(\Omega). \qquad (10)$$

The Airy stress function v can be expressed in the form

$$v = -E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) - E_0(\frac{1}{2}\Phi(u,u) + Lu)$$

and we reformulate the system (**??**) ,(**??**) into the following variational inequality:

Problem $\mathcal{P}$. *Find* $u \in \mathcal{K}$ *such that* $\dot{u} \in L_2(I; H^2(\Omega))$ *and the inequality*

$$\begin{aligned}
&\textstyle\int_Q (E_1 A(\dot{u}, y - u) + E_0 A(u, y - u))\, dx\, dt \\
&+ \textstyle\int_Q [u, E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u) + Lu)](y - u)\, dx\, dt \\
&+ \textstyle\int_Q \Delta_k \left(E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u) + Lu)\right)(y - u)\, dx\, dt \qquad (11)\\
&- \textstyle\int_Q (a\nabla\dot{u} \cdot \nabla(\dot{y} - \dot{u}) + \dot{u}(\dot{y} - \dot{u}))\, dx\, dt + \int_\Omega (a\nabla\dot{u} \cdot \nabla(y - u) + \dot{u}(y - u))\,(T,\cdot)\, dx \\
&\geq \textstyle\int_\Omega (a\nabla u_1 \cdot \nabla(y(0,\cdot) - u_0) + u_1(y(0,\cdot) - u_0))\, dx + \int_Q f(y_1 - u)\, dx\, dt
\end{aligned}$$

is satisfied for any $y \in \mathcal{K}$.

For any $\eta > 0$ we define the *penalized problem*
Problem $\mathcal{P}_\eta$. *Find* $u \in H^{1,2}(Q)$ *such that* $\dot{u} \in L_2(I; H^2(\Omega))$, $\ddot{u} \in L_2(I; H^1(\Omega))$, the equation

$$\begin{aligned} &\textstyle\int_Q \big(\ddot{u}z + a\nabla\ddot{u}\cdot\nabla z + E_1 A(\dot{u}, z) + E_0 A(u, z)\big)\,dx\,dt \\ &+ \textstyle\int_Q [u, E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u) + Lu)]z\,dx\,dt \\ &+ \textstyle\int_Q \Delta_k\,\big(E_1\partial_t(\frac{1}{2}\Phi(u,u) + Lu) + E_0(\frac{1}{2}\Phi(u,u) + Lu)\big)\,z\,dx\,dt \\ &= \textstyle\int_Q (f + \eta^{-1}u^-)z\,dx\,dt, \end{aligned} \tag{12}$$

holds for any $z \in L_2(I; H^2(\Omega))$ and the conditions (**??**) remain valid.

Applying the Galerkin method, we obtain in a similar way as in [**?**] the existence and uniqueness of a solution to the penalized problem with the *a priori* estimates

$$\begin{aligned} &\|\dot{u}\|^2_{L_2(I;H^2(\Omega))} + \|\dot{u}\|^2_{L_\infty(I;H^1(\Omega))} + \|u\|^2_{L_\infty(I;H^2(\Omega))} \\ &+\|\partial_t\Phi(u,u)\|^2_{L_2(I;H^2(\Omega))} + \|\partial_t Lu\|^2_{L_2(I;H^2(\Omega))} \le c \equiv c(f, u_0, u_1). \end{aligned} \tag{13}$$

Moreover the estimates (**??**), (**??**) imply

$$\|\partial_t\Phi(u,u)\|_{L_2(I;W^2_p(\Omega))} + \|\partial_t Lu\|_{L_2(I;W^2_p(\Omega))} \le c_p \equiv c_p(f, u_0, u_1)\ \forall\, p > 2. \tag{14}$$

The estimates are obviously η independent. Since for a fixed $\eta > 0$ the penalty term $\eta^{-1}u^-$ belongs to $H^1(Q)$, this together with (**??**) and (**??**) yields an estimate of $\ddot{u} - a\Delta\ddot{u}$ in $L_2(I; H^2(\Omega)^*)$. Applying the *a priori* estimates of solutions to the penalized problem we obtain

Theorem 2.1 *Let* $f \in L_2(Q)$, $u_i \in H^2(\Omega)$, $i = 0,\ 1$. *Then there exists a solution* $u \in H^{1,2}(Q)$ *of the contact Problem* $\mathcal{P}$.

Proof. We perform the limit process $\eta \searrow 0$ and write u_η for the solution of the problem $\mathcal{P}_\eta$. To get the crucial estimate for the penalty, we put $z = 1$ in (**??**). We get

$$\int_Q \eta^{-1}u_\eta^-\,dx\,dt = \int_Q (\ddot{u}_\eta - f)\,dx\,dt = \int_Q (\dot{u}_\eta(T,\cdot) - u_1)\,dx - \int_Q f\,dx\,dt$$

and the estimate

$$\|\eta^{-1}u_\eta^-\|_{L_1(Q)} \le c(f, u_0, u_1). \tag{15}$$

which is independent of η. The standard imbedding $H^2(\Omega) \hookrightarrow L_1(\Omega)$ and the *a priori* estimates (**??**) and (**??**) imply for the functional φ_η given as

$$\varphi_\eta : w \mapsto \int_Q a\nabla\dot{u}_\eta\nabla w + \dot{u}_\eta w\,dx\,dt \tag{16}$$

the estimate $\|\dot{\varphi}_\eta\|_{L_1(I;H^2(\Omega)^*)} \le c$.

Applying Lemma 1.1 we obtain that the system $\{\varphi_\eta; \eta > 0\}$ is relatively compact in $L_2(I; H^1(\Omega)^*)$.

The *a priori* estimates (**??**), (**??**), the last relative compactness and the standard theory of linear elliptic equations yield the existence of a sequence $\eta_k \searrow 0$ such that for $u_k \equiv u_{\eta_k}$ the following convergence hold for any real $p \ge 1$:

$$\begin{aligned} &\dot{u}_k \rightharpoonup \dot{u} \text{ in } L_2(I; H^2(\Omega)) \\ &\dot{u}_k \to \dot{u} \text{ in } L_2(I; W^1_p(\Omega)), \\ &u_k \to u \text{ in } C(I; W^1_p(\Omega)), \\ &\tfrac{1}{2}\partial_t\Phi(u_k, u_k) + \partial_t Lu_k \rightharpoonup \tfrac{1}{2}\partial_t\Phi(u,u) + \partial_t Lu \text{ in } L_2(I; W^2_p(\Omega)). \end{aligned} \tag{17}$$

The crucial strong convergence of the derivatives is the consequence of the relative compactness of $\{\varphi_\eta; \eta > 0\}$ and of the first weak convergence in (**??**) (see [**?**]) for details. Inserting the test function $z = y - u_k$ in (**??**) for $y \in \mathcal{K}$, performing the integration by parts in the terms containing $\ddot{u}$, applying the convergence (**??**) and the weak lower semicontinuity verifies that the limit u is a solution of the original problem $\mathcal{P}$.

Remark 2.2 *The initial-boundary value problem for a dynamic contact of a clamped shell with Dirichlet zero boundary for deflections can be formulated and solved analogously as in the case of the viscoelastic plate in [**?**].*

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