(1) Riešte okrajovú úlohu

$x^2u'' + 2xu' = x,\ 1 < x < 2;\ u(1) - u'(1) = u(2) = 0.$

Samoadjungovaný tvar : $(x^2u')' = x$,

$x^2u' = \int x\,dx = \frac{1}{2}x^2 + c_1.$

$$u'(x) = \frac{1}{2} + \frac{c_1}{x^2}.$$

$$u(x) = \int u'(x)\,dx = \frac{1}{2}x - c_1\frac{1}{x} + c_2,$$

$u(2) = 1 - c_1\frac{1}{2} + c_2 = 0$

$u(1) - u'(1) = \frac{1}{2} - c_1 + c_2 - (\frac{1}{2} + c_1) = c_2 - 2c_1 = 0$

$c_2 = 2c_1\ \frac{3}{2}c_1 + 1 = 0 \Longrightarrow c_1 = -\frac{2}{3},\ c_2 = -\frac{4}{3},$

$$\underline{u(x) = \frac{1}{2}x + \frac{2}{3x} - \frac{4}{3}.}$$

(2) Riešte okrajovú úlohu :

$-u'' + u = \sin x,\ u'(0) = u(\pi) = 0.$

$r^2 - 1 = 0,\ r_{1,2} = \pm 1.$

$u_h(x) = c_1 \cosh x + c_2 \sinh x,$

$u_p(x) = A\cos x + B\sin x,$

$u_p''(x) = -A\cos x - B\sin x.$

$-u_p''(x) + 4u_p(x) = 2A\cos x + 2B\sin x = \sin x \Longrightarrow$

$A = 0,\ B = \frac{1}{2} \Longrightarrow u_p(x) = \frac{1}{2}\sin x,$

$u(x) = u_h(x) + u_p(x) = c_1\cosh x + c_2\sinh x + \frac{1}{2}\sin x,$

$u'(x) = c_1\sinh x + c_2\cosh x + \frac{1}{2}\cos x,$

$u'(0) = c_2 + \frac{1}{2} = 0 \Longrightarrow c_2 = -\frac{1}{2};$

$u(\pi) = c_1\cosh\pi - \frac{1}{2}sinh\pi = 0 \Longrightarrow c_1 = \frac{\sinh\pi}{2\cosh\pi}$

$$\underline{u(x) = \frac{\sinh\pi}{2\cosh\pi}\cosh x - \frac{1}{2}\sinh x + \frac{1}{2}\sin x.}$$

(3) Rozložte funkciu $f(x) = 2x$ do Fourierovho radu podľa ortogonálneho systému $\{\sin \frac{(2n-1)\pi}{2} x\}$ v priestore $L_2(0,1)$.

$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{(2n-1)\pi}{2} x,$

$c_n = \frac{2}{\int}{}_0^1 2x \sin \frac{(2n-1)\pi}{2} x \, dx$

$= 2 \left(\left[- \frac{4x \cos \frac{(2n-1)\pi}{2} x}{(2n-1)\pi} \right]_0^1 + \int_0^1 \frac{4 \cos \frac{(2n-1)\pi}{2} x}{(2n-1)\pi} \, dx \right)$

$= 0 + \frac{16}{(2n-1)^2 \pi^2} \sin \left(\frac{(2n-1)\pi}{2} \right) = \frac{16(-1)^{n-1}}{(2n-1)^2 \pi^2}.$

$$\underline{u(x) = \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{(2n-1)^2 \pi^2} \sin \frac{(2n-1)\pi}{2} x.}$$