(1) Riešte okrajovú úlohu

$e^{-x}u'' - e^{-x}u' = e^x, \; 0 < x < 1; \; u'(0) = u(1) = 0.$

Samoadjungovaný tvar : $(e^{-x}u')' = e^x$,

$e^{-x}u' = \int e^x \, dx = e^x + c_1.$

$$u'(x) = e^{2x} + c_1 e^x.$$

$u'(0) = 1 + c_1 = 0 \; \Rightarrow \; c_1 = -1.$

$$u'(x) = e^{2x} - e^x.$$

$$u(x) = \int u'(x) \, dx = \frac{1}{2}e^{2x} - e^x + c_2,$$

$u(1) = \frac{1}{2}e^2 - e + c_2 = 0 \; \Rightarrow \; c_2 = e - \frac{1}{2}e^2,$

$$\underline{u(x) = \frac{1}{2}e^{2x} - e^x + e - \frac{1}{2}e^2.}$$

(2) Riešte okrajovú úlohu :

$-u'' + 4u = \cos 2x, \; u'(0) = u'(\frac{\pi}{2}) = 0.$

$r^2 - 4 = 0, \; r_{1,2} = \pm 2.$

$u_h(x) = c_1 \cosh 2x + c_2 \sinh 2x,$

$u_p(x) = A \cos 2x + B \sin 2x,$

$u_p''(x) = -4A \cos 2x - 4B \sin 2x.$

$-u_p''(x) + 4u_p(x) = 8A \cos 2x + 8B \sin 2x = \cos 2x \Longrightarrow$

$A = \frac{1}{8}, \; B = 0 \Longrightarrow u_p(x) = \frac{1}{8} \cos 2x,$

$u(x) = u_h(x) + u_p(x) = c_1 \cosh 2x + c_2 \sinh 2x + \frac{1}{8} \cos 2x,$

$u'(x) = -2c_1 \sinh 2x + 2c_2 \cosh 2x - \frac{1}{4} \sin 2x,$

$u'(0) = 2c_2 = 0 \Longrightarrow c_2 = 0;$

$u'(\frac{\pi}{2}) = -2c_1 \sinh \pi = 0 \Longrightarrow c_1 = 0$

$$\underline{u(x) = \frac{1}{8} \cos 2x.}$$

(3) Rozložte funkciu $f(x) = x$ do Fourierovho radu podľa ortogonálneho systému $\{\cos(2n-1)x\}$ v priestore $L_2(0, \frac{\pi}{2})$.

$f(x) = \sum_{n=1}^{\infty} c_n \cos(2n-1)x,$

$$c_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} x \cos(2n-1)x \, dx$$
$$= \frac{4}{\pi} \left(\left[\frac{x \sin(2n-1)x}{2n-1} \right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{2n-1} \, dx \right)$$
$$= \frac{4}{\pi} \left(\frac{\pi(-1)^{n-1}}{2(2n-1)}) - \frac{1}{(2n-1)^2} \right).$$

$$u(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \left(\frac{\pi(-1)^{n-1}}{2(2n-1)} - \frac{1}{(2n-1)^2} \right) \cos(2n-1)x.$$