(1) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ 0 < y < 2,$

$u(0,y) = \frac{\partial u}{\partial x}(\pi, y) = u(x,0) = 0,\ u(x,2) = x.$

$u(x,y) = X(x)Y(y),$

$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda,$

$Y''(y) - \lambda Y(y) = 0.$

$X''(x) + \lambda X(x) = 0,\ X(0) = X'(\pi) = 0,$

$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$X(0) = c_1 = 0,\ c_2 = 1,\ X(x) = \sin\sqrt{\lambda}x$

$X'(\pi) = \sqrt{\lambda}\cos\pi\sqrt{\lambda} = 0,$

$\pi\sqrt{\lambda} = (2n-1)\frac{\pi}{2},\ \sqrt{\lambda} = \frac{2n-1}{2},\ n \in N,$

$\lambda \equiv \lambda_n = \left(\frac{2n-1}{2}\right)^2,$

$X(x) \equiv X_n(x) = \sin\frac{2n-1}{2}x,\ n \in N.$

$Y_n''(y) - \left(\frac{2n-1}{2}\right)^2 Y_n(y) = 0,\ n \in N,$

$Y_n(y) = a_n \cosh\frac{2n-1}{2}y + b_n \sinh\frac{2n-1}{2}y.$

$Y_n(0) = a_n = 0,\ \Rightarrow\ Y_n(y) = b_n \sinh\frac{2n-1}{2}y,\ n \in N.$

$u_n(x,y) = Y_n(y)X_n(x) = b_n \sinh\frac{2n-1}{2}y\ \sin\frac{2n-1}{2}x,$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} b_n \sinh\frac{2n-1}{2}y\ \sin\frac{2n-1}{2}x.$$

$x = u(x,2) = \sum_{n=1}^{\infty} b_n \sinh(2n-1)\ \sin\frac{2n-1}{2}x.$

$B_n = b_n \sinh(2n-1) = \frac{2}{\pi}\int_0^{\pi} x \sin\frac{2n-1}{2}x\, dx =$

$\frac{2}{\pi}\left(\left[x\frac{-2\cos\frac{2n-1}{2}x}{2n-1}\right]_0^{\pi} + \int_0^{\pi}\frac{2\cos\frac{2n-1}{2}x}{2n-1}\, dx\right) = \frac{8(-1)^{n-1}}{\pi(2n-1)^2}.$

$$b_n = \frac{B_n}{\sinh(2n-1)} = \frac{8(-1)^{n-1}}{\pi(2n-1)^2\sinh(2n-1)},\ n \in N.$$

$$u(x,y) = \sum_{n=1}^{\infty}\frac{8(-1)^{n-1}}{\pi(2n-1)^2\sinh(2n-1)}\sinh\frac{2n-1}{2}y\ \sin\frac{2n-1}{2}x.$$

(2) Riešte okrajovú úlohu

$$\Delta u(x,y) = 0,\ x^2 + y^2 < 1,\ x > 0,\ y > 0,$$
$$u(x,0) = \frac{\partial u}{\partial x}(0,y) = 0,\ \frac{\partial u}{\partial \vec{n}}(x,y) = 1,\ \text{ak } x^2 + y^2 = 1,$$

$$r^2 \Delta u(r,\varphi) = r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0,$$
$$u(r,0) = \frac{\partial u}{\partial \varphi}(r, \tfrac{\pi}{2}) = 0,\ \frac{\partial u}{\partial r}(1,\varphi) = 1,\ 0 < r < 1,\ 0 < \varphi < \tfrac{\pi}{2}.$$

$$u(r,\varphi) = R(r)\Phi(\varphi),$$

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda,$$

$$r^2 R'' + rR' - \lambda R = 0,\ 0 \le r < 1.$$

$$\Phi'' + \lambda \Phi = 0,\ 0 < \varphi < \tfrac{\pi}{2};\ \Phi(0) = \Phi'(\tfrac{\pi}{2}) = 0.$$

$$\lambda > 0,\ \Phi(\varphi) = c_1 \cos(\sqrt{\lambda}\varphi) + c_2 \sin(\sqrt{\lambda}\varphi),$$
$$\Phi(0) = c_1 = 0 \Rightarrow c_1 = 0,\ c_2 = 1$$
$$\Phi'(\tfrac{\pi}{2}) = \cos(\sqrt{\lambda}\tfrac{\pi}{2}) = 0 \Rightarrow \sqrt{\lambda}\tfrac{\pi}{2} = (2n-1)\tfrac{\pi}{2},\ n \in N$$

$$\lambda \equiv \lambda_n = (2n-1)^2,\ \Phi_n(\varphi) = \sin(2n-1)\varphi,\ n \in N$$

$$r^2 R_n'' + rR_n' - (2n-1)^2 R_n = 0,\ 0 \le r < 1,$$

$$R_n(r) = a_n r^{2n-1} + b_n r^{-(2n-1)},\ b_n = 0,\ n \in N$$
$$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_n r^{2n-1} \sin(2n-1)\varphi,$$

$$u(r,\varphi) = \sum_{n=1}^{\infty} u_n(r,\varphi) = \sum_{n=1}^{\infty} a_n r^{2n-1} \sin(2n-1)\varphi,$$

$$1 = \frac{\partial u}{\partial r}(1,\varphi) = \sum_{n=1}^{\infty} a_n(2n-1) \sin(2n-1)\varphi,$$

$$A_n = a_n(2n-1) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin(2n-1)\varphi \, d\varphi = \frac{4}{(2n-1)\pi},$$

$$a_n = \frac{A_n}{2n-1} = \frac{4}{(2n-1)^2\pi},\ n \in N.$$

$$\underline{u(r,\varphi) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi} r^{2n-1} \sin(2n-1)\varphi.}$$

(3) Riešte začiatočno-okrajovú úlohu

$\frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial x^2} = 0,\ t > 0,\ 0 < x < 1;$
$\frac{\partial u}{\partial x}(t,0) = u(t,1) = 0,\ u(0,x) = x - 1.$

$u(t,x) = T(t)X(x),\ \frac{T'}{4T} = \frac{X''}{X} = -\lambda,$
$T' + 4\lambda T(t) = 0,$
$X'' + \lambda X,\ X'(0) = X(1) = 0,$
$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$X'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0,\ c_1 = 1$
$X(x) = \cos\sqrt{\lambda}x,$
$X(1) = \cos\sqrt{\lambda}1 = 0 \Rightarrow \sqrt{\lambda} = (2n-1)\frac{\pi}{2},\ n = 1, 2, ...$
$\lambda \equiv \lambda_n = \left[\frac{(2n-1)\pi}{2}\right]^2,\ X_n(x) = \cos\frac{(2n-1)\pi}{2}x,\ n = 1, 2, ...$
$T_n' + 4\left[\frac{(2n-1)\pi}{2}\right]^2 T_n(t) = 0 \Rightarrow$
$T_n(t) = a_n e^{-(2n-1)^2\pi^2 t},\ n = 1, 2, ...$

$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2\pi^2 t} \cos\frac{(2n-1)\pi}{2}x,$

$x - 1 = u(0,x) = \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi}{2}x,$

$a_n = 2\int_0^1 (x-1)\cos\frac{(2n-1)\pi}{2}x\,dx$
$= 2\left(\left[(x-1)\frac{2\sin\frac{(2n-1)\pi}{2}x}{(2n-1)\pi}\right]_0^1 - \int_0^1 \frac{2\sin\frac{(2n-1)\pi}{2}x}{\pi(2n-1)}\,dx\right)$

$= \frac{8}{\pi^2(2n-1)^2}.$

$$u(t,x) = \sum_{n=1}^{\infty} \frac{8}{\pi^2(2n-1)^2} e^{-(2n-1)^2\pi^2 t} \cos\frac{(2n-1)\pi}{2}x.$$

(4) Riešte začiatočno-okrajovú úlohu

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,\ t > 0,\ 0 < x < \pi;$

$u(t, 0) = u(t, \pi) = 0,\ u(0, x) = x^2 - \pi x.$

$u(t, x) = T(t)X(x),\ \frac{T'}{T} = \frac{X''}{X} = -\lambda,$

$T' + \lambda T(t) = 0,$

$X'' + \lambda X,\ X(0) = X(\pi) = 0,$

$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$X(0) = c_1 = 0 \Rightarrow c_2 = 1$

$X(x) = \sin\sqrt{\lambda}x,$

$X(\pi) = \sin\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = n\pi,\ n = 1, 2, ...$

$\lambda \equiv \lambda_n = n^2,\ X_n(x) = \sin nx,\ n = 1, 2, ...$

$T_n' + n^2 T_n(t) = 0 \Rightarrow T_n(t) = a_n e^{-n^2 t},\ n = 1, 2, ...$

$u(t, x) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx,$

$x^2 - \pi x = u(0, x) = \sum_{n=1}^{\infty} a_n \sin nx,$

$a_n = \frac{2}{\pi} \int_0^\pi (x^2 - \pi x) \sin nx\, dx$

$= \frac{2}{\pi} \left(\left[(x^2 - \pi x) \frac{-\cos nx}{n} \right]_0^\pi + \int_0^\pi (2x - \pi) \frac{\cos nx}{n}\, dx \right)$

$= \frac{2}{\pi} \left(\left[(2x - \pi) \frac{\sin nx}{n^2} \right]_0^\pi - \int_0^\pi \frac{2 \sin nx}{n^2}\, dx \right)$

$= \frac{4}{\pi} \left[\frac{\cos nx}{n^3} \right]_0^\pi = \frac{4[(-1)^n - 1]}{\pi n^3}.$

$$\underline{u(t, x) = \sum_{n=1}^{\infty} \frac{4[(-1)^n - 1]}{\pi n^3} e^{-n^2 t} \sin nx.}$$