(1) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ 0 < y < 2,$
$u(0,y) = \frac{\partial u}{\partial x}(\pi, y) = 0,\ \frac{\partial u}{\partial y}(x,0) = 0,\ u(x,2) = x.$

$u(x,y) = X(x)Y(y),$
$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$
$Y''(y) - \lambda Y(y) = 0.$

$X''(x) + \lambda X(x) = 0,\ X(0) = X'(\pi) = 0,$
$\lambda > 0,\ X(x) = \sin\sqrt{\lambda}x.$

$X'(\pi) = \sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = (2n-1)\frac{\pi}{2} \Rightarrow \sqrt{\lambda} = \frac{2n-1}{2}$
$\lambda \equiv \lambda_n = \left(\frac{2n-1}{2}\right)^2,\ X(x) \equiv X_n(x) = \sin\frac{2n-1}{2}x,\ n \in N.$

$Y_n''(y) - \left(\frac{2n-1}{2}\right)^2 Y_n(y) = 0,\ n \in N,$
$Y_n(y) = a_n \cosh\frac{2n-1}{2}y + b_n \sinh\frac{2n-1}{2}y.$
$Y_n'(0) = b_n\frac{2n-1}{2} = 0\ \Rightarrow\ Y_n(y) = a_n\cosh\frac{2n-1}{2}y.$

$u_n(x,y) = Y_n(y)X_n(x) = a_n\cosh\frac{2n-1}{2}y\sin\frac{2n-1}{2}x,$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} a_n\cosh\frac{2n-1}{2}y\sin\frac{2n-1}{2}x.$$

$x = u(x,2) = \sum_{n=1}^{\infty} a_n\cosh(2n-1)\sin\frac{2n-1}{2}x.$

$A_n = a_n\cosh(2n-1) = \frac{2}{\pi}\int_0^{\pi} x\sin\frac{2n-1}{2}x\,dx = \frac{8(-1)^{n-1}}{\pi(2n-1)^2},$

$$a_n = \frac{A_n}{\cosh(2n-1)} = \frac{8(-1)^{n-1}}{\pi(2n-1)^2\cosh(2n-1)},\ n \in N.$$

$$\underline{u(x,y) = \sum_{n=1}^{\infty}\frac{8(-1)^{n-1}}{\pi(2n-1)^2\cosh(2n-1)}\cosh(2n-1)y\sin\frac{2n-1}{2}x.}$$

(2) Riešte okrajovú úlohu

$\Delta u(x,y) = 0,\ x^2 + y^2 < 4,\ x > 0,\ y > 0$
$u(x,0) = u(0,y) = 0,\ \frac{\partial u}{\partial \vec{n}} = 1,\ \text{ak } x^2 + y^2 = 4$

$r^2\Delta u(r,\varphi) = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0,\ 0 < r < 2,\ 0 < \varphi < \frac{\pi}{2}.$
$u(r,\varphi) = R(r)\Phi(\varphi),$
$\frac{r^2R'' + rR'}{R(r)} = -\frac{\Phi''}{\Phi(\varphi)} = \lambda$
$r^2R'' + rR' - \lambda R = 0,\ 0 < r < 2.$
$\Phi'' + \lambda\Phi = 0,\ 0 < \varphi < \frac{\pi}{2};\ \Phi(0) = \Phi(\frac{\pi}{2}) = 0.$
$\lambda > 0,\ \Phi(\varphi) = \sin(\sqrt{\lambda}\varphi),$
$\Phi(\frac{\pi}{2}) = \sin(\sqrt{\lambda}\frac{\pi}{2}) = 0 \Rightarrow \sqrt{\lambda}\frac{\pi}{2} = n\pi,$
$\lambda \equiv \lambda_n = (2n)^2,\ \Phi_n(\varphi) = \sin 2n\varphi,\ \ n \in N.$
$r^2R_n'' + rR_n' - (2n)^2R_n = 0,\ 0 \le r < 2 \Rightarrow R_n(r) = a_nr^{2n},$

$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_nr^{2n}\sin 2n\varphi,,\ n \in N$

$$u(r,\varphi) = \sum_{n=1}^{\infty} u_n(r,\varphi) = \sum_{n=1}^{\infty} a_n r^{2n} \sin 2n\varphi.$$

$1 = \frac{\partial u}{\partial r}(2,\varphi) = \sum_{n=1}^{\infty} a_n 2n 2^{2n-1} \sin 2n\varphi,$
$A_n = 2n2^{2n-1} a_n = \frac{4}{\pi}\int_0^{\pi} \sin 2n\varphi \, d\varphi = \frac{2}{n\pi}[1-(-1)^n],$

$$a_n = \frac{A_n}{2n2^{2n-1}} = \frac{2[1-(-1)^n]}{\pi 2^{2n} n^2}, \ n \in N.$$

$$\underline{u(r,\varphi) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{\pi 2^{2n} n^2} r^{2n} \sin 2n\varphi = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{\pi n^2} \left(\frac{r}{2}\right)^{2n} \sin 2n\varphi.}$$

(3) Riešte úlohu na vlastné hodnoty:
$\Delta u + \lambda u(x,y) = 0, \ u < x < 2\pi, \ 0 < y < 1,$
$u(0,y) = u(2\pi,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0.$

$u(x,y) = X(x)Y(y), \ \lambda = \mu + \nu,$

$X'' + \mu X(x) = 0, \ X(0) = X(2\pi) = 0,$
$\mu_m = \left(\frac{m}{2}\right)^2, \ X_m(x) = \sin \frac{m}{2}x, \ m \in N.$

$Y'' + \nu Y(y) = 0, \ Y'(0) = Y'(1) = 0,$
$\nu_n = (n\pi)^2, \ Y(y) = \cos n\pi \, y, \ n \in N_0.$

$$\underline{\lambda_{m,n} = \left(\frac{m}{2}\right)^2 + (n\pi)^2, \ u_{m,n} = \sin \frac{m}{2} x \cos n\pi \, y, \ m \in N, \, n \in N_0.}$$

(4) Riešte začiatočno-okrajovú úlohu
$\frac{\partial u}{\partial t} - 3\frac{\partial^2 u}{\partial x^2} = 0, \ \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \ u(0,x) = x.$

$u(t,x) = T(t)X(x),$
$\frac{T'}{3T} = \frac{X''}{X} = -\lambda,$

$T' + 3\lambda T(t) = 0,$

$X'' + \lambda X, \ X'(0) = X'(1) = 0,$
$\lambda \geq 0, \ \lambda_0 = 0, \ X_0(x) = 1,$
$\lambda > 0, \ X(x) = \cos\sqrt{\lambda}x,$
$X'(1) = -\sqrt{\lambda}\sin\sqrt{\lambda}1 = 0 \Rightarrow \sqrt{\lambda} = n\pi, \ n = 1,2,...$
$\lambda \equiv \lambda_n = n^2\pi^2, \ X_n(x) = \cos n\pi x, \ n = 0,1,2,...$

$T_n' + 3n^2\pi^2 T(t) = 0 \Rightarrow T_n(t) = a_n e^{-3n^2\pi^2 t}, \ n = 0,1,2,...$

$u(t,x) = \sum_{n=0}^{\infty} a_n e^{-3n^2\pi^2 t} \cos n\pi x,$
$x = u(0,x) = \sum_{n=0}^{\infty} a_n \cos n\pi x,$
$a_0 = \int_0^1 x\,dx = \frac{1}{2}, \ a_n = 2\int_0^1 x \cos n\pi x \, dx = \frac{2[(-1)^n - 1]}{n^2\pi^2}.$

$$\underline{u(t,x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2} e^{-3n^2\pi^2 t} \cos n\pi x.}$$