(1) Riešte okrajovú úlohu

$e^{-x}u'' - e^{-x}u' = e^x,\ 0 < x < 1;\ u(0) - u'(0) = u(1) = 0.$

Samoadjungovaný tvar : $(e^{-x}u')' = e^x$,

$e^{-x}u' = \int e^x\,dx = e^x + c_1.$

$$u'(x) = e^x(e^x + c_1) = e^{2x} + c_1e^x.$$

$$u(x) = \int u'(x)\,dx = \int (e^{2x} + c_1e^x)\,dx = \frac{e^{2x}}{2} + c_1e^x + c_2.$$

$u(0) - u'(0) = \frac{1}{2} + c_1 + c_2 - 1 - c_1 = -\frac{1}{2} + c_2 = 0 \ \Rightarrow\ c_2 = -\frac{1}{2}$,

$u(1) = \frac{e^2}{2} + c_1e - \frac{1}{2} = 0 \ \Rightarrow\ c_1 = -\frac{e}{2} + \frac{1}{2e}$,

$$\underline{u(x) = \frac{1}{2}\left[e^{2x} + \left(\frac{1}{e} - e\right)e^x - 1\right].}$$

(2) Riešte úlohu na vlastné hodnoty a vlastné funkcie:

$u'' + \lambda u = 0,\ u(0) = u(2\pi) = 0.$

$\lambda > 0$, pretože sa rieši Dirichletova okrajová úloha.

$r^2 + \lambda = 0,\ r_{1,2} = \pm i\sqrt{\lambda}.$

$u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$u(0) = c_1 \cos 0 = c_1 = 0 \ \Rightarrow\ c_1 = 0,\ c_2 = 1.$

$u(2\pi) = \sin\sqrt{\lambda}2\pi = 0 \ \Rightarrow\ \sqrt{\lambda}2\pi = n\pi,$

$\sqrt{\lambda} = \frac{n}{2},$

$$\underline{\lambda \equiv \lambda_n = \left(\frac{n}{2}\right)^2,\ u(x) \equiv u_n(x) = \sin\frac{n}{2}x,\ n \in N.}$$

(3) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 2\pi,\ 0 < y < 1,$
$u(0,y) = u(2\pi, y) = 0,\ u(x,0) = 0,\ u(x,1) = 2.$

$u(x,y) = X(x)Y(y),$
$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$
$Y''(y) - \lambda Y(y) = 0.$

$X''(x) + \lambda X(x) = 0,\ X(0) = X(2\pi) = 0,$
$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$

Z Príkladu 2:

$\lambda \equiv \lambda_n = \left(\frac{n}{2}\right)^2,\ X(x) \equiv X_n(x) = \sin\frac{n}{2}x,\ n \in N.$

$Y_n''(y) - \left(\frac{n}{2}\right)^2 Y_n(y) = 0,\ n \in N,$
$Y_n(y) = a_n \cosh\frac{n}{2}y + b_n \sinh\frac{n}{2}y.$
$Y_n(0) = a_n = 0,\ \Rightarrow\ Y_n(y) = b_n \sinh\frac{n}{2}y.$

$u_n(x,y) = Y_n(y)X_n(x) = b_n \sinh\frac{n}{2}y \sin\frac{n}{2}x,$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} b_n \sinh\frac{n}{2}y \sin\frac{n}{2}x.$$

$2 = u(x,1) = \sum_{n=1}^{\infty} b_n \sinh n \sin\frac{n}{2}x.$
$B_n = b_n \sinh n = \frac{1}{\pi}\int_0^{2\pi} 2\sin\frac{n}{2}x\,dx$
$= \frac{4}{n\pi}(\cos 0 - \cos n\pi) = \frac{4}{n\pi}[1-(-1)^n],$

$$b_n = \frac{B_n}{\sinh n} = \frac{4[1-(-1)^n]}{n\pi \sinh n},\ n \in N.$$

$$\underline{u(x,y) = \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n\pi \sinh n} \sinh\frac{n}{2}y \sin\frac{n}{2}x.}$$