(1) Riešte okrajovú úlohu

$x^2u'' + 2xu' = x,\ 1 < x < 2;\ u'(1) = u(2) = 0.$

Samoadjungovaný tvar : $(x^2u')' = x$,

$x^2u' = \int x\,dx = \frac{x^2}{2} + c_1.$

$$u'(x) = x^{-2}(\frac{x^2}{2} + c_1) = \frac{1}{2} + \frac{c_1}{x^2}.$$

$u'(1) = \frac{1}{2} + c_1 = 0 \ \Rightarrow\ c_1 = -\frac{1}{2}.$

$$u'(x) = \frac{1}{2} - \frac{1}{2x^2}.$$

$$u(x) = \int u'(x)\,dx = \int (\frac{1}{2} - \frac{1}{2x^2})\,dx = \frac{x}{2} + \frac{1}{2x} + c_2,$$

$u(2) = 1 + \frac{1}{4} + c_2 = 0 \ \Rightarrow\ c_2 = -\frac{5}{4},$

$$\underline{u(x) = \frac{x}{2} + \frac{1}{2x} - \frac{5}{4}.}$$

(2) Riešte úlohu na vlastné hodnoty a vlastné funkcie:

$u'' + \lambda u = 0,\ u'(0) = u(\pi) = 0.$

$\lambda > 0$, pretože sa rieši zmiešaná okrajová a <u>nie</u> Neumannova okrajová úloha.

$r^2 + \lambda = 0,\ r_{1,2} = \pm i\sqrt{\lambda}.$

$u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x,$

$u'(0) = c_2\sqrt{\lambda}\cos 0 = c_2\sqrt{\lambda} = 0 \ \Rightarrow\ c_2 = 0,\ c_1 = 1.$

$u(\pi) = \cos\sqrt{\lambda}\pi = 0 \ \Rightarrow\ \sqrt{\lambda}\pi = (2n-1)\frac{\pi}{2},$

$\sqrt{\lambda} = \frac{2n-1}{2},$

$$\underline{\lambda \equiv \lambda_n = \left(\frac{2n-1}{2}\right)^2,\ u(x) \equiv u_n(x) = \cos\frac{2n-1}{2}x,\ n \in N.}$$

(3) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ 0 < y < 1,$

$\frac{\partial u}{\partial x}(0, y) = u(\pi, y) = 0,\ u(x, 0) = 0,\ u(x, 1) = 1.$

$u(x, y) = X(x)Y(y),$

$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$

$Y''(y) - \lambda Y(y) = 0.$

$X''(x) + \lambda X(x) = 0,\ X'(0) = X(\pi) = 0,$

$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

Z Príkladu 2:

$\lambda \equiv \lambda_n = \left(\frac{2n-1}{2}\right)^2,\ X(x) \equiv X_n(x) = \cos\frac{2n-1}{2}x,\ n \in N.$

$Y_n''(y) - \left(\frac{2n-1}{2}\right)^2 Y_n(y) = 0,\ n \in N,$

$Y_n(y) = a_n \cosh\frac{2n-1}{2}y + b_n \sinh\frac{2n-1}{2}y.$

$Y_n(0) = a_n = 0,\ \Rightarrow\ Y_n(y) = b_n \sinh\frac{2n-1}{2}y.$

$u_n(x, y) = Y_n(y)X_n(x) = b_n \sinh\frac{2n-1}{2}y\cos\frac{2n-1}{2}x,$

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} b_n \sinh\frac{2n-1}{2}y\cos\frac{2n-1}{2}x.$$

$1 = u(x, 1) = \sum_{n=1}^{\infty} b_n \sinh\frac{2n-1}{2}\cos\frac{2n-1}{2}x.$

$B_n = b_n \sinh\frac{2n-1}{2} = \frac{2}{\pi}\int_0^{\pi} 1\cos\frac{2n-1}{2}x\,dx$

$= \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} = \frac{4(-1)^{n-1}}{(2n-1)\pi}.$

$$b_n = \frac{B_n}{\sinh\frac{2n-1}{2}} = \frac{4(-1)^{n-1}}{(2n-1)\pi\sinh\frac{2n-1}{2}},\ n \in N.$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{(2n-1)\pi\sinh\frac{2n-1}{2}}\sinh\frac{2n-1}{2}y\cos\frac{2n-1}{2}x.$$