(1) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 2,\ 0 < y < 1,$

$\frac{\partial u}{\partial y}(x,0) = u(x,1) = u(0,y) = 0,\ \frac{\partial u}{\partial x}(2,y) = 1.$

$u(x,y) = X(x)Y(y),$
$X''(x) - \lambda X(x) = 0$
$Y''(y) + \lambda Y(y) = 0,\ Y'(0) = Y(1) = 0.$
$\lambda > 0,\ Y(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y,$
$Y'(y) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}y + c_2\sqrt{\lambda}\cos\sqrt{\lambda}y,$
$Y'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0,\ c_1 = 1,$
$Y(y) = \cos\sqrt{\lambda}y,\ Y(1) = \cos\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = \frac{(2k-1)\pi}{2},$
$\lambda \equiv \lambda_k = \frac{(2k-1)^2\pi^2}{4},\ Y(y) \equiv Y_k(y) = \cos\frac{(2k-1)\pi}{2}y,\ k = 1, 2, ...$

$X_k'' - \lambda_k X_k(x) = 0,\ X_k'' - \frac{(2k-1)^2\pi^2}{4}X_k(x) = 0$
$X_k(x) = a_k \cosh\frac{(2k-1)\pi}{2}x + b_k \sinh\frac{(2k-1)\pi}{2}x.$
$X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh\frac{(2k-1)\pi}{2}x.$

$u_k(x,y) = X_k(x)Y_k(y) = b_k \sinh\frac{(2k-1)\pi}{2}x \cos\frac{(2k-1)\pi}{2}y,$
$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} b_k \sinh\frac{(2k-1)\pi}{2}x \cos\frac{(2k-1)\pi}{2}y.$
$1 = \frac{\partial u}{\partial y}(2,y) = \sum_{k=1}^{\infty} b_k \frac{(2k-1)\pi}{2}\cosh(2k-1)\pi \cos\frac{(2k-1)\pi}{2}y.$
$B_k = b_k\frac{(2k-1)\pi}{2}\cosh(2k-1)\pi = 2\int_0^1 1\cos\frac{(2k-1)\pi}{2}y\,dy$
$= \frac{4}{(2k-1)\pi}\sin\frac{(2k-1)\pi}{2} = \frac{4(-1)^{k-1}}{(2k-1)\pi}.$
$b_k = \frac{B_k}{\frac{(2k-1)\pi}{2}\cosh(2k-1)\pi} = \frac{8(-1)^{k-1}}{(2k-1)^2\pi^2\cosh(2k-1)\pi}.$

$\underline{u(x,y) = \sum_{k=1}^{\infty} \frac{8(-1)^{k-1}}{(2k-1)^2\pi^2\cosh(2k-1)\pi}\sinh\frac{(2k-1)\pi}{2}x\cos\frac{(2k-1)\pi}{2}y.}$

(2) Riešte okrajovú úlohu

$\triangle u(x,y) = 1,\ x^2 + y^2 < 1,\ u(x,y) = 0,$ ak $x^2 + y^2 = 1.$
$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 1,\ 0 < r < 1.$
$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = r^2,\ 0 \le r < 1,\ 0 \le \phi \le 2\pi,\ u(1,\phi) = 0$
$u(r,\phi) = R(r),\ 0 \le r < 1,\ R(1) = 0.$
$r^2R''(r) + rR'(r) = r^2,\ rR'' + R' = r,$
$(rR'(r))' = r,\ R(1) = 0,$
$rR'(r) = \frac{1}{2}r^2 + C,\ R'(r) = \frac{1}{2}r + \frac{C_1}{r},$
$R(r) = \frac{1}{4}r^2 + C_1\ln r + C_2,\ C_1 = 0,$ pretože riešenie je ohraničené.
$R(1) = \frac{1}{4} + C_2 = 0 \Rightarrow C_2 = -\frac{1}{4},$
$R(r) = u(r,\phi) = \frac{1}{4}r^2 - \frac{1}{4} = \frac{1}{4}(r^2 - 1),$
$\underline{u(x,y) = \frac{1}{4}(x^2 + y^2 - 1).}$

(3) Riešte začiatočno-okrajovú úlohu

$\frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial x^2} = 0,\ t > 0,\ 0 < x < \pi,$
$u(0,x) = x,\ u(t,0) = u(t,\pi) = 0.$
$u(t,x) = T(t)X(x),\ \frac{T'(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\lambda$
$T'(t) + 4\lambda T(t) = 0,\ t > 0,$
$X''(x) + \lambda X(x) = 0,\ X(0) = X(\pi) = 0,\ \lambda > 0.$
$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$
$X(0) = c_1 = 0,\ c_2 = 1,\ X(x) = \sin\sqrt{\lambda}x,$
$X(\pi) = \sin\sqrt{\lambda}\pi = 0, \Rightarrow \sqrt{\lambda}\pi = k\pi,$
$\lambda \equiv \lambda_k = k^2,\ X \equiv X_k(x) = \sin kx,\ k = 1, 2, ...$

$T_k'(t) + 4\lambda_k T_k(t) = T_k'(t) + 4k^2 T_k(t) = 0,\ t > 0,$
$T_k(t) = a_k e^{-4k^2 t},\ t > 0,$

$u_k(t,x) = T_k(t)X_k(x) = a_k e^{-4k^2 t} \sin kx,$
$u(t,x) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} a_k e^{-4k^2 t} \sin kx,$
$x = u(0,x) = \sum_{k=1}^{\infty} a_k \sin kx,$
$a_k = \frac{2}{\pi}\int_0^{\pi} x \sin kx\, dx = \frac{2}{\pi}\{\left[x\frac{(-\cos kx)}{k}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos kx}{k}\, dx\} = \frac{2(-1)^{k-1}}{k}.$

$\underline{u(t,x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{k} e^{-4k^2 t} \sin kx.}$

(4) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 2,\ 0 < y < \pi,$
$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,\pi) = u(0,y) = 0,\ u(2,y) = 1.$

$u(x,y) = X(x)Y(y),$
$X''(x) - \lambda X(x) = 0$
$Y''(y) + \lambda Y(y) = 0,\ Y'(0) = Y'(1) = 0.$

a) $\lambda_0 = 0,\ Y''(y) = 0 \Rightarrow Y_0(y) = c_1 y + c_2.$
$Y_0'(0) = Y_0'(\pi) = c_1 = 0,\ c_2 = 1 \Rightarrow Y_0(y) = 1.$

b) $\lambda > 0,\quad Y(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y,$
$Y'(y) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}y + c_2\sqrt{\lambda}\cos\sqrt{\lambda}y,$
$Y'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0,\ c_1 = 1,$
$Y(y) = \cos\sqrt{\lambda}y,\ Y'(\pi) = -\sin\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = k\pi,$
$\lambda \equiv \lambda_k = k^2,\ Y(y) \equiv Y_k(y) = \cos ky,\ k = 0, 1, 2, ...$

$X_k'' - \lambda_k X_k(x) = 0,\ X_k'' - k^2 X_k(x) = 0,\ k = 0, 1, 2, ...$
$X_0(x) = a_0 + b_0 x,\ X_0(0) = a_0 = 0 \Rightarrow X_0(x) = b_0 x,$
$X_k(x) = a_k \cosh kx + b_k \sinh kx.$
$X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh kx,\ k = 1, 2, ...$

$u_0(x,y) = X_0(x)Y_0(y) = b_0 x,$
$u_k(x,y) = X_k(x)Y_k(y) = b_k \sinh kx \cos ky,\ \ k = 1,2,...$
$u(x,y) = \sum_{k=0}^{\infty} u_k(x,y) = b_0 x + \sum_{k=1}^{\infty} b_k \sinh kx \cos ky.$
$1 = u(2,y) = 2b_0 + \sum_{k=1}^{\infty} b_k \sinh 2k \cos ky.$
$B_0 = 2b_0 = \frac{\int_0^\pi 1^2\,dy}{\int_0^\pi 1^2\,dy} = 1 \Rightarrow b_0 = \frac{1}{2},$
$B_k = b_k \sinh 2k = \frac{2}{\pi}\int_0^\pi 1 \cos ky\,dy = 0 \Rightarrow\ \ b_k = 0,\ \ k = 1,2,...$

$\underline{u(x,y) = \frac{1}{2}x.}$

(5) Riešte okrajovú úlohu

$\triangle u(x,y) = 0,\ \ x^2+y^2 < 1,\ \ y > 0,\ \ u(x,y) = 1,$ ak $x^2+y^2 = 1.$
$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = 0,\ \ 0 \le r < 1,\ \ 0 < \phi < \pi,$
$u(r,0) = u(r,\pi) = 0,\ \ u(1,\phi) = 1.$
$u(r,\phi) = R(r)\Phi(\phi),$
$\frac{r^2R''(r)+rR'(r)}{R(r)} = -\frac{\Phi''(\phi)}{\Phi(\phi)} = \lambda.$
$r^2R''(r) + rR'(r) - \lambda R(r) = 0,\ \ 0 \le r < 1,$
$\Phi''(\phi) + \lambda\Phi(\phi) = 0,\ \ 0 < \phi < \pi,\ \ \Phi(0) = \Phi(\pi) = 0,$
$\lambda > 0,\ \ \Phi(\phi) = c_1 \cos\sqrt{\lambda}\phi + c_2 \sin\sqrt{\lambda}\phi,$
$\Phi(0) = c_1 = 0,\ \ c_2 = 1, \rightarrow \Phi(\phi) = \sin\sqrt{\lambda}\phi,$
$\Phi(\pi) = \sin\sqrt{\lambda}\pi = 0, \Rightarrow \sqrt{\lambda}\pi = k\pi,$
$\lambda \equiv \lambda_k = k^2,\ \ \Phi(\phi) \equiv \Phi_k(\phi) = \sin k\phi,$
$r^2R''(r) + rR'(r) - k^2R(r) = 0,\ \ 0 \le r < 1,$
$R(r) = a_k r^k + b_k r^{-k},\ \ k = 1,2,...$
$b_k = 0$, pretože riešenie je ohraničené.
$R(r) \equiv R_k(r) = a_k r^k,\ \ k = 1,2,...,0 < r < 1,$
$u_k(r,\phi) = R_k(r)\Phi_k(\phi) = a_k r^k \sin k\phi,\ \ k = 1,2,...$
$u(r,\phi) = \sum_{k=1}^{\infty} u_k(r,\phi) = \sum_{k=1}^{\infty} a_k r^k \sin k\phi.$
$1 = u(1,\phi) = \sum_{k=1}^{\infty} a_k \sin k\phi.$
$a_k = \frac{2}{\pi}\int_0^\pi \sin k\phi\,d\phi = \frac{2}{\pi}\left[\frac{-\cos k\phi}{k}\right]_0^\pi = \frac{2}{k\pi}[1-(-1)^k],$

$\underline{u(r,\phi) = \sum_{k=1}^{\infty} \frac{2}{k\pi}[1-(-1)^k] r^k \sin k\phi.}$

(6) Riešte začiatočno-okrajovú úlohu

$\frac{\partial u}{\partial t} - 9\frac{\partial^2 u}{\partial x^2} = 0,\ \ t > 0,\ \ 0 < x < 2,$
$u(0,x) = x,\ \ u(t,0) = u(t,2) = 0.$
$u(t,x) = T(t)X(x),\ \ \frac{T'(t)}{9T(t)} = \frac{X''(x)}{X(x)} = -\lambda$
$T'(t) + 9\lambda T(t) = 0,\ \ t > 0,$
$X''(x) + \lambda X(x) = 0,\ \ X(0) = X(2) = 0,\ \ \lambda > 0.$
$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$
$X(0) = c_1 = 0,\ \ c_2 = 1,\ \ X(x) = \sin\sqrt{\lambda}x,$

$X(2) = \sin\sqrt{\lambda}2 = 0 \;\Rightarrow 2\sqrt{\lambda} = k\pi,$
$\lambda \equiv \lambda_k = \frac{k^2\pi^2}{4},\; X \equiv X_k(x) = \sin\frac{k\pi}{2}x,\; k = 1, 2, ...$

$T_k'(t) + 9\lambda_k T_k(t) = T_k'(t) + 9\frac{k^2\pi^2}{4}T_k(t) = 0,\; t > 0,$
$T_k(t) = a_k e^{-\frac{9k^2\pi^2}{4}t},\; t > 0,$

$u_k(t,x) = T_k(t)X_k(x) = a_k e^{-\frac{9k^2\pi^2}{4}t}\sin\frac{k\pi}{2}x,$
$u(t,x) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} a_k e^{-\frac{9k^2\pi^2}{4}t}\sin\frac{k\pi}{2}x,$
$x = u(0,x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{2}x,$
$a_k = \frac{2}{\pi}\int_0^2 x\sin\frac{k\pi}{2}x\,dx =$
$\frac{2}{\pi}\{\big[x(-\frac{2}{k\pi})\cos\frac{k\pi}{2}x\big]_0^2 + \int_0^{\pi}\frac{2}{k\pi}\cos\frac{k\pi}{2}x\,dx\} = \frac{4(-1)^{k-1}}{k\pi}.$

$\underline{u(t,x) = \sum_{k=1}^{\infty}\frac{4(-1)^{k-1}}{k\pi}e^{-\frac{9k^2\pi^2}{4}t}\sin\frac{k\pi}{2}x,\; t > 0,\; 0 < x < 2.}$