(1) Riešte okrajovú úlohu

$\Delta u(x,y)=0,\ x^2+y^2<1,$
$y>0;\ u(x,0)=0,\ \frac{\partial u}{\partial \vec{n}}=2,\ \text{ak } x^2+y^2=1$

$r^2\Delta u(r,\varphi)=\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \varphi^2}=0,\ 0<r<1,\ 0<\varphi<\pi.$
$u(r,\varphi)=R(r)\Phi(\varphi),$
$\frac{r^2R''+rR'}{R(r)}=-\frac{\Phi''}{\Phi(\varphi)}=\lambda$
$r^2R''+rR'-\lambda R=0,\ 0<r<1.$
$\Phi''+\lambda\Phi=0,\ 0<\varphi<\pi;\ \Phi(0)=\Phi(\pi)=0.$
$\lambda>0,\ \Phi(\varphi)=sin(\sqrt{\lambda}\varphi),$
$\Phi(\pi)=\sin(\sqrt{\lambda}\pi)=0\Rightarrow\sqrt{\lambda}\pi=n\pi,\ n=1,2,...,$
$\lambda\equiv\lambda_n=n^2,\ \Phi_n(\varphi)=\sin n\varphi,\ n=1,2,..$
$r^2R_n''+rR_n'-n^2R_n=0,\ 0\le r<1;\Rightarrow R_n(r)=a_nr^n,\ n=1,2,....$
$u(r,\varphi)=\sum_{n=1}^{\infty}a_nr^n\sin n\varphi,$
$\frac{\partial u}{\partial r}(1,\varphi)=\sum_{n=1}^{\infty}na_n\sin n\varphi=2,$
$na_n=\frac{2}{\pi}\int_0^{\pi}2\sin n\varphi\,d\varphi=\frac{4}{n\pi}[1-(-1)^n],$
$u(r,\varphi)=\sum_{n=1}^{\infty}\frac{4}{n^2\pi}[1-(-1)^n]r^n\sin n\varphi.$

(2) Riešte úlohu na vlastné hodnoty:

$\Delta u+\lambda u(x,y)=0,\ u<x<\pi,\ 0<y<2,$
$u(0,y)=u(\pi,y)=\frac{\partial u}{\partial y}(x,0)=u(x,2)=0.$

$u(x,y)=X(x)Y(y),\ \lambda=\mu+\nu,$
$X''+\mu X(x)=0,\ X(0)=X(\pi)=0,\ \mu_m=m^2,\ X_m(x)=\sin mx.$
$Y''+\nu Y(y)=0,\ Y'(0)=Y(2)=0,\ \nu_n=\frac{(2n-1)^2\pi^2}{16},\ Y(y)=\cos\frac{(2n-1)\pi}{4}y,$
$\lambda_{m,n}=m^2+\frac{(2n-1)^2\pi^2}{16},\ u_{m,n}=\sin mx\cos\frac{(2n-1)\pi}{4}y,\ m,n=1,2,....$

(3) Riešte začiatočno-okrajovú úlohu

$\frac{\partial u}{\partial t}-3\frac{\partial^2 u}{\partial x^2}=0,\ \frac{\partial u}{\partial x}(t,0)=\frac{\partial u}{\partial x}(t,1)=0,\ u(0,x)=x.$

$u(t,x)=T(t)X(x),\ \frac{T'}{3T}=\frac{X''}{X}=-\lambda,$
$T'+3\lambda T(t)=0,$
$X''+\lambda X,\ X'(0)=X'(1)=0,$
$\lambda\ge 0,\ \lambda_0=0,\ X_0(x)=1,$
$\lambda>0,\ X(x)=\cos\sqrt{\lambda}x,$
$X'(1)=-\sqrt{\lambda}\sin\sqrt{\lambda}1=0\Rightarrow\sqrt{\lambda}=n\pi,\ n=1,2,...$
$\lambda\equiv\lambda_n=n^2\pi^2,\ X_n(x)=\cos n\pi x,\ n=0,1,2,...$
$T_n'+3n^2\pi^2T(t)=0\Rightarrow T_n(t)=a_ne^{-3n^2\pi^2t},\ n=0,1,2,...$
$u(t,x)=\sum_{n=0}^{\infty}a_ne^{-3n^2\pi^2t}\cos n\pi x,$
$x=u(0,x)=\sum_{n=0}^{\infty}a_n\cos n\pi x,$
$a_0=\int_0^1x\,dx=\frac{1}{2},\ a_n=2\int_0^1x\cos n\pi x\,dx=\frac{2[(-1)^n-1]}{n^2\pi^2}.$
$u(t,x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{n^2\pi^2}e^{-3n^2\pi^2t}\cos n\pi x.$