

(1) Riešte okrajovú úlohu

$$\cos^2 x \, u'' - \sin 2x \, u' = \cos x, \quad 0 < x < \frac{\pi}{4}, \quad u'(0) = 0, \quad u\left(\frac{\pi}{4}\right) = 1,$$

$$(\cos^2 x)' = -2 \sin x \cos x = -\sin 2x.$$

$$(\cos^2 x \, u')' = \cos x, \quad \cos^2 x \, u' = \sin x + c_1,$$

$$u' = \frac{\sin x}{\cos^2 x} + \frac{c_1}{\cos^2 x},$$

$$u(x) = \frac{1}{\cos x} - c_1 \tan x + c_2,$$

$$u'(0) = c_1 = 0, \quad u\left(\frac{\pi}{4}\right) = \sqrt{2} + c_2 = 1 \Rightarrow c_2 = 1 - \sqrt{2},$$

$$u(x) = \frac{1}{\cos x} + 1 - \sqrt{2}.$$

(2) Riešte okrajovú úlohu

$$-u'' + 4u = x, \quad u(0) = 0, \quad u(1) + u'(1) = 1.$$

$$u(x) = c_1 \cosh 2x + c_2 \sinh 2x + \frac{1}{4}x, \quad u'(x) = 2c_1 \sinh 2x + 2c_2 \cosh 2x + \frac{1}{4},$$

$$u(0) = c_1 = 0,$$

$$u(1) + u'(1) = c_2 \sinh 2 + \frac{1}{4} + 2c_2 \cosh 2 + \frac{1}{4} = 1,$$

$$c_2 = \frac{1}{2(2 \cosh 2 + \sinh 2)},$$

$$u(x) = \frac{\sinh 2x}{2(2 \cosh 2 + \sinh 2)} + \frac{1}{4}x.$$

(3) Riešte úlohu na vlastné hodnoty a vlastné funkcie.

a) $u'' + \lambda u = 0, \quad 0 < x < \frac{1}{2}, \quad u(0) = u\left(\frac{\pi}{2}\right) = 0.$

$$\lambda > 0, \quad u(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x,$$

$$u(0) = c_1 = 0, \quad c_2 = 1 \Rightarrow u(x) = \sin \sqrt{\lambda}x,$$

$$u\left(\frac{\pi}{2}\right) = \sin \sqrt{\lambda} \frac{\pi}{2} = 0 \Rightarrow 2\sqrt{\lambda} \frac{\pi}{2} = n\pi, \quad \sqrt{\lambda} = 2n, \quad n = 1, 2, \dots$$

$$\lambda_n = 4n^2, \quad u_n(x) = \sin 2nx.$$

b) $u'' + \lambda u = 0, \quad 0 < x < 2, \quad u'(0) = u'\left(\frac{\pi}{2}\right) = 0.$

$$\lambda_0 = 0, \quad u_0(x) = c_1 + c_2x,$$

$$u'(0) = u'\left(\frac{\pi}{2}\right) = c_2 = 0, \quad c_1 = 1, \Rightarrow u_0(x) = 1,$$

$$\lambda > 0, \quad u(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x,$$

$$u'(x) = -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}x,$$

$$u'(0) = \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0, \quad c_1 = 1, \quad u(x) = \cos \sqrt{\lambda}x,$$

$$u'\left(\frac{\pi}{2}\right) = -\sqrt{\lambda} \sin \sqrt{\lambda} \frac{\pi}{2} = 0 \Rightarrow \sqrt{\lambda} \frac{\pi}{2} = n\pi, \quad \sqrt{\lambda} = 2n.$$

$$\lambda_n = n^2, \quad u_n(x) = \cos 2nx, \quad n = 0, 1, 2, \dots$$