

(1) Riešte okrajovú úlohu

$$\sin^2 x u'' + \sin 2x u' = \sin x, \quad \frac{\pi}{4} < x < \frac{\pi}{2}, \quad u\left(\frac{\pi}{4}\right) = 1, \quad u'\left(\frac{\pi}{2}\right) = 0.$$

$$(\sin^2 x)' = 2 \sin x \cos x = \sin 2x.$$

$$(\sin^2 x u')' = \sin x, \quad \sin^2 x u' = -\cos x + c_1,$$

$$u' = -\frac{\cos x}{\sin^2 x} + \frac{c_1}{\sin^2 x},$$

$$u(x) = \frac{1}{\sin x} - c_1 \cot x + c_2,$$

$$u'\left(\frac{\pi}{2}\right) = c_1 = 0, \quad u\left(\frac{\pi}{4}\right) = \sqrt{2} + c_2 = 1 \Rightarrow c_2 = 1 - \sqrt{2},$$

$$u(x) = \frac{1}{\sin x} + 1 - \sqrt{2}.$$

(2) Riešte okrajovú úlohu

$$u'' - 9u = 0, \quad u'(0) = 0, \quad u(2) + u'(2) = 1.$$

$$u(x) = c_1 \cosh 3x + c_2 \sinh 3x, \quad u'(x) = 3c_1 \sinh 3x + 3c_2 \cosh 3x,$$

$$u'(0) = 3c_2 = 0 \Rightarrow c_2 = 0,$$

$$u(2) + u'(2) = c_1 \cosh 6 + 3c_1 \sinh 6 = 1,$$

$$c_1 = \frac{1}{\cosh 6 + 3 \sinh 6},$$

$$u(x) = \frac{\cosh 3x}{\cosh 6 + 3 \sinh 6}.$$

(3) Riešte úlohu na vlastné hodnoty a vlastné funkcie.

a) $u'' + \lambda u = 0, \quad 0 < x < 2, \quad u(0) = u'(2) = 0.$

$$\lambda > 0, \quad u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

$$u(0) = c_1 = 0, \quad c_2 = 1 \Rightarrow u(x) = \sin \sqrt{\lambda} x,$$

$$u'(2) = \sqrt{\lambda} \cos 2\sqrt{\lambda} = 0 \Rightarrow 2\sqrt{\lambda} = (2n-1)\frac{\pi}{2}, \quad \sqrt{\lambda} = \frac{(2n-1)\pi}{4},$$

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{16}, \quad u_n(x) = \sin \frac{(2n-1)\pi}{4} x.$$

b) $u'' + \lambda u = 0, \quad 0 < x < 2, \quad u'(0) = u(2) = 0.$

$$\lambda > 0, \quad u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

$$u'(x) = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x,$$

$$u'(0) = \sqrt{\lambda} c_2 = 0 \Rightarrow c_2 = 0, \quad c_1 = 1, \quad u(x) = \cos \sqrt{\lambda} x,$$

$$u(2) = \cos \sqrt{\lambda} 2 = 0 \Rightarrow 2\sqrt{\lambda} = (2n-1)\frac{\pi}{2}, \quad \sqrt{\lambda} = \frac{(2n-1)\pi}{4}.$$

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{16}, \quad u_n(x) = \cos \frac{(2n-1)\pi}{4} x.$$