(1) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 2,\ 0 < y < 1,$

$u(0, y) = \frac{\partial u}{\partial x}(2, y) = u(x, 0) = 0,\ u(x, 1) = x.$

$u(x, y) = X(x)Y(y),$

$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda,$

$Y''(y) - \lambda Y(y) = 0.$

$X''(x) + \lambda X(x) = 0,\ X(0) = X'(2) = 0,$

$\lambda > 0,\ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$X(0) = c_1 = 0,\ c_2 = 1,\ X(x) = \sin\sqrt{\lambda}x$

$X'(2) = \sqrt{\lambda}\cos 2\sqrt{\lambda} = 0,$

$2\sqrt{\lambda} = (2n-1)\frac{\pi}{2},\ \sqrt{\lambda} = (2n-1)\frac{\pi}{4},\ n \in N,$

$\lambda \equiv \lambda_n = \left[\frac{(2n-1)\pi}{4}\right]^2,$

$X(x) \equiv X_n(x) = \sin\frac{(2n-1)\pi}{4}x,\ n \in N.$

$Y_n''(y) - \left[\frac{(2n-1)\pi}{4}\right]^2 Y_n(y) = 0,\ n \in N,$

$Y_n(y) = a_n \cosh\frac{(2n-1)\pi}{4}y + b_n \sinh\frac{(2n-1)\pi}{4}y.$

$Y_n(0) = a_n = 0,\ \Rightarrow\ Y_n(y) = b_n \sinh\frac{(2n-1)\pi}{4}y,\ n \in N.$

$u_n(x, y) = Y_n(y)X_n(x) = b_n \sinh\frac{(2n-1)\pi}{4}y\ \sin\frac{(2n-1)\pi}{4}x,$

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} b_n \sinh\frac{(2n-1)\pi}{4}y\ \sin\frac{(2n-1)\pi}{4}x.$$

$x = u(x, 1) = \sum_{n=1}^{\infty} b_n \sinh\frac{(2n-1)\pi}{4}\ \sin\frac{(2n-1)\pi}{4}x.$

$B_n = b_n \sinh\frac{(2n-1)\pi}{4} = \int_0^2 x \sin\frac{(2n-1)\pi}{4}x\,dx =$

$\left[x\frac{-4\cos\frac{(2n-1)\pi}{4}x}{(2n-1)\pi}\right]_0^2 + \int_0^2 \frac{4\cos\frac{(2n-1)\pi}{4}x}{(2n-1)\pi}\,dx = \frac{16(-1)^{n-1}}{(2n-1)^2\pi^2}.$

$$b_n = \frac{B_n}{\sinh\frac{(2n-1)\pi}{4}} = \frac{16(-1)^{n-1}}{(2n-1)^2\pi^2\sinh\frac{(2n-1)\pi}{4}},\ n \in N.$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{(2n-1)^2\pi^2\sinh\frac{(2n-1)\pi}{4}} \sinh\frac{(2n-1)\pi}{4}y\ \sin\frac{(2n-1)\pi}{4}x.$$

(2) Riešte okrajovú úlohu

$$\Delta u(x,y) = 0,\ 1 < x^2 + y^2 < 4,\ y > 0,$$
$$u(x,0) = 0,\ u(x,y) = 0,\ \text{ak } x^2 + y^2 = 1,$$
$$u(x,y) = 1,\ \text{ak } x^2 + y^2 = 4.$$

$$r^2 \Delta u(r,\varphi) = r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0,\ 1 < r < 2,\ 0 < \varphi < \pi.$$

$$u(r,\varphi) = R(r)\Phi(\varphi),$$
$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda,$$
$$r^2 R'' + rR' - \lambda R = 0,\ 1 < r < 2.$$
$$\Phi'' + \lambda\Phi = 0,\ 0 < \varphi < \pi;\ \Phi(0) = \Phi(\pi) = 0.$$
$$\lambda > 0,\ \Phi(\varphi) = c_1 \cos(\sqrt{\lambda}\varphi) + c_2 \sin(\sqrt{\lambda}\varphi),$$
$$\Phi(0) = c_1 = 0 \Rightarrow c_1 = 0,\ c_2 = 1$$
$$\Phi(\pi) = \sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = n\pi,\ n \in N$$

$$\lambda \equiv \lambda_n = n^2,\ \Phi_n(\varphi) = \sin n\varphi,\ n \in N$$

$$r^2 R_n'' + rR_n' - n^2 R_n = 0,\ 1 < r < 2,\ R_n(1) = 0;$$

$$R_n(r) = a_n r^n + b_n r^{-n},\ n \in N$$
$$R_n(1) = a_n + b_n = 0 \Rightarrow b_n = -a_n,$$
$$R_n(r) = a_n(r^n - r^{-n}),\ n \in N.$$
$$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_n(r^n - r^{-n}) \sin n\varphi,$$

$$u(r,\varphi) = \sum_{n=1}^{\infty} u_n(r,\varphi) = \sum_{n=1}^{\infty} a_n(r^n - r^{-n}) \sin n\varphi,$$

$$1 = u(2,\varphi) = \sum_{n=1}^{\infty} na_n(2^n - 2^{-n}) \sin n\varphi,$$
$$A_n = a_n(2^n - 2^{-n}) = \frac{2}{\pi}\int_0^{\pi} \sin n\varphi \, d\varphi = \frac{2[1-(-1)^n]}{n\pi},$$
$$a_n = \frac{A_n}{2^n - 2^{-n}} = \frac{2[1-(-1)^n]}{n\pi(2^n - 2^{-n})},\ n \in N.$$
$$\underline{u(r,\varphi) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \frac{r^n - r^{-n}}{2^n - 2^{-n}} \sin n\varphi.}$$