(1) Riešte okrajovú úlohu

$u'' - 4u = 2xe^{-x},\ u(0) = 0,\ u'(1) = 2.$

$u(x) = c_1 \cosh 2x + c_2 \sinh 2x + u_p(x),\ u_p(x) = (ax + b)e^{-x},$

$u_p(x)'' = 0\, e^{-x} - 2a\, e^{-x} + (ax + b)e^{-x} = (ax + b - 2a)e^{-x},$

$(ax+b-2a)e^{-x}-4(ax+b)e^{-x} = (-3ax-2a-3b)e^{-x} = 2xe^{-x},$

$$a = -\frac{2}{3},\ b = \frac{4}{9}$$

$$u(x) = c_1 \cosh 2x + c_2 \sinh 2x + (-\frac{2}{3}x + \frac{4}{9})e^{-x}$$

$u(0) = c_1 + \frac{4}{9} = 0 \Rightarrow c_1 = -\frac{4}{9}.$

$u'(1) = 2c_1 \sinh 2 + 2c_2 \cosh 2 + (\frac{2}{3} - \frac{4}{9} - \frac{2}{3})e^{-1}$
$= -\frac{8}{9} \sinh 2 + 2c_2 \cosh 2 + \frac{4}{9}e^{-1} = 2$

$$c_2 = \frac{9 + 4 \sinh 2 - 2e^{-1}}{9 \cosh 2}$$

$$u(x) = -\frac{4}{9} \cosh 2x + \frac{9 + 4 \sinh 2 - 2e^{-1}}{9 \cosh 2} \sinh 2x + (-\frac{2}{3}x + \frac{4}{9})e^{-x}$$

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(2) Riešte okrajovú úlohu

$-(1 - x^2)u'' + 2xu' = 1,\ 0 < x < \frac{1}{2},\ u'(0) = u(\frac{1}{2}) = 0.$

$[(1 - x^2)u']' = -1 \Rightarrow (1 - x^2)u' = -x + c_1\ \Rightarrow u'(x) = \frac{-x+c_1}{1-x^2}.$

$u'(0) = c_1 = 0\ \Rightarrow\ u'(x) = \frac{-x}{1-x^2},$

$u(x) = \int \frac{-x}{1-x^2}\, dx = \frac{1}{2}\, \ln(1 - x^2) + c_2,$

$u(\frac{1}{2}) = \frac{1}{2}\, \ln(\frac{3}{4}) + c_2 = 0\ \Rightarrow\ c_2 = -\frac{1}{2}\, \ln(\frac{3}{4})$

$$u(x) = \frac{1}{2}\, \ln(1 - x^2) - \frac{1}{2}\, \ln(\frac{3}{4}) = \ln \sqrt{\frac{4(1 - x^2)}{3}}.$$

(3) Riešte úlohu na vlastné hodnoty a vlastné funkcie.

a) $u'' + \lambda u = 0,\ 0 < x < \frac{1}{2},\ u(0) = u'(\frac{\pi}{2}) = 0.$

$\lambda > 0,\ u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$
$u(0) = c_1 = 0,\ c_2 = 1\ \Rightarrow u(x) = \sin\sqrt{\lambda}x,$
$u'(\frac{\pi}{2}) = \sqrt{\lambda}\cos\sqrt{\lambda}\frac{\pi}{2} = 0 \Rightarrow \sqrt{\lambda}\frac{\pi}{2} = (2n-1)\frac{\pi}{2},\ \sqrt{\lambda} = 2n-1.$

$\lambda_n = (2n-1)^2,\ u_n(x) = \sin(2n-1)x,\ n \in \mathbb{N}.$

b) $u'' + \lambda u = 0,\ 0 < x < 2,\ u'(0) = u(\frac{\pi}{2}) = 0.$

$\lambda > 0,\ u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$
$u'(0) = \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0,\ c_1 = 1,\ u(x) = \cos\sqrt{\lambda}x,$
$u(\frac{\pi}{2}) = \cos\sqrt{\lambda}\frac{\pi}{2} = 0 \Rightarrow \sqrt{\lambda}\frac{\pi}{2} = (2n-1)\frac{\pi}{2},\ \sqrt{\lambda} = 2n-1,$

$\lambda_n = (2n-1)^2,\ u_n(x) = \cos(2n-1)x,\ n \in \mathbb{N}.$