Boundary Value Problems for Elliptic Equations:

1. Solve the boundary value problems using the Fourier method

   $\triangle u = 0,\ 0 < x < 1,\ 0 < y < 1,$

   a) $u(0,y) = \frac{\partial u}{\partial x}(1,y) = \ u(x,0) = 0,\ u(x,1) = x$

   b) $u(0,y) = \frac{\partial u}{\partial x}(1,y) = \ u(x,0) = 0,\ \frac{\partial u}{\partial y}(x,1) = x$

   c) $\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = u(1,y) = 0,\ u(0,y) = y^2$

   d) $u(x,0) = u(x,1) = 0,\ u(0,y) = \sin \pi y,\ \frac{\partial u}{\partial x}(1,y) = 1.$

2. Solve the boundary value problems using the Fourier method

   $\triangle u = 0,\ 0 < x < 2,\ 0 < y < \pi,$

   a) $u(x,0) = \frac{\partial u}{\partial y}(x,\pi) = \ u(0,y) = 0,\ u(2,y) = y$

   b) $\frac{\partial u}{\partial y} u(x,0) = u(x,\pi) = \ u(0,y) = 0,\ \frac{\partial u}{\partial x}(2,y) = y$

   c) $\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(2,y) = u(x,\pi) = 0,\ u(x,0) = \sin x$

   d) $u(0,y) = u(2,y) = 0,\ u(x,0) = u(x,\pi) = x^2 - 2x.$

3. Solve the boundary value problems using the polar coordinates

   $\triangle u = 0,\ r^2 = x^2 + y^2 < 4.$

   a) $u(2,\phi) = 1$

   b)$u(2,\phi) = \phi^2 - 2\pi\phi$

   c)$u(2,\phi) = |\sin\phi|$

4. Solve the boundary value problems using the polar coordinates

   $\triangle u = 0,\ x^2 + y^2 < 1,\ y > 0.$

   a) $u(x,0) = 0,\ u(x,\sqrt{1-x^2}) = 1 - x^2$

   b) $u(x,0) = 0,\ \frac{\partial u}{\partial \vec{n}}(x,\sqrt{1-x^2}) = 1,$

   with $\vec{n} = \vec{r}$ the unique outer normal vector .

5. Solve the boundary value problems

   $\triangle u = 0,\ x^2 + y^2 < 4,\ x > 0,\ y > 0.$

   a) $u(x,0) = u(0,y) = 0,\ u(x,\sqrt{4-x^2}) = x (= 2\cos\phi)$

b) $u(x,0) = \frac{\partial u}{\partial x}(0,y) = 0,\ u(x,\sqrt{4-x^2}) = 1$

6. Solve the boundary value problems

   $\triangle u = 0,\ 1 < x^2 + y^2 < 4,\ y > 0.$

   a) $u(x,0) = 0,\ u(x,\sqrt{1-x^2}) = 1,\ u(x,\sqrt{4-x^2}) = 2$

   b) $u(x,0) = 0,\ \ u(x,\sqrt{1-x^2}) = 1,\ \frac{\partial u}{\partial \vec{n}}(x,\sqrt{4-x^2}) = 1$

7. Solve the eigenvalue and eigenfunction problem

$$\triangle u(x,y) + \lambda u(x,y) = 0,\ 0 < x < \pi,\ 0 < y < 2,$$

   with boundary conditions

   a) $u(0,y) = u(\pi,y) = u(x,0) = u(x,2) = 0,$

   b) $u(0,y) = u(\pi,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,2) = 0,$

   c) $u(0,y) = \frac{\partial u}{\partial x}(\pi,y) = \frac{\partial u}{\partial y}(x,0) = u(x,2) = 0,$

8. Solve the boundary value problems

$$-\triangle u = f(x,y),\ (x,y) \in \Omega$$

   for the deflection of a squared membrane $\Omega = (0,1) \times (0,1)$, if

   a) $f(x,y) = 1,\ u|_{\partial\Omega} = 0,$

   b) $f(x,y) = 1,\ u(0,y) = u(1,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0,$

   c) $f(x,y) = xy,\ u(0,y) = u(1,y) = u(x,0) = \frac{\partial u}{\partial y}(x,1) = 0.$

   Apply the double Fourier series with respect to eigenfunctions of the problem $\triangle u + \lambda u = 0$ with the same boundary conditions.

9. Solve the boundary value problems

   $-\triangle u = f(x,y),\ (x,y) \in \Omega$

   for the deflection of the rectangular membrane $\Omega = (0,\pi) \times (0,1)$, if

   a) $f(x,y) = x,\ u|_{\partial\Omega} = 0,$

   b) $f(x,y) = y,\ u(0,y) = u(\pi,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0,$

   c) $f(x,y) = y \sin x,\ u(0,y) = u(\pi,y) = u(x,0) = \frac{\partial u}{\partial y}(x,1) = 0.$