

(1) Riešte okrajovú úlohu

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 1,$$

$$\frac{\partial u}{\partial y}(x, 0) = u(x, 1) = u(0, y) = 0, \quad \frac{\partial u}{\partial x}(2, y) = 1.$$

$$u(x, y) = X(x)Y(y),$$

$$X''(x) - \lambda X(x) = 0$$

$$Y'''(y) + \lambda Y(y) = 0, \quad Y'(0) = Y(1) = 0.$$

$$\lambda > 0, \quad Y(y) = c_1 \cos \sqrt{\lambda} y + c_2 \sin \sqrt{\lambda} y,$$

$$Y'(y) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} y + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} y,$$

$$Y'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0, \quad c_1 = 1,$$

$$Y(y) = \cos \sqrt{\lambda} y, \quad Y(1) = \cos \sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = \frac{(2k-1)\pi}{2},$$

$$\lambda \equiv \lambda_k = \frac{(2k-1)^2 \pi^2}{4}, \quad Y(y) \equiv Y_k(y) = \cos \frac{(2k-1)\pi}{2} y, \quad k = 1, 2, \dots$$

$$X_k'' - \lambda_k X_k(x) = 0, \quad X_k'' - \frac{(2k-1)^2 \pi^2}{4} X_k(x) = 0$$

$$X_k(x) = a_k \cosh \frac{(2k-1)\pi}{2} x + b_k \sinh \frac{(2k-1)\pi}{2} x.$$

$$X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh \frac{(2k-1)\pi}{2} x.$$

$$u_k(x, y) = X_k(x)Y_k(y) = b_k \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y,$$

$$u(x, y) = \sum_{k=1}^{\infty} u_k(x, y) = \sum_{k=1}^{\infty} b_k \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y.$$

$$1 = \frac{\partial u}{\partial y}(2, y) = \sum_{k=1}^{\infty} b_k \frac{(2k-1)\pi}{2} \cosh(2k-1)\pi \cos \frac{(2k-1)\pi}{2} y.$$

$$B_k = b_k \frac{(2k-1)\pi}{2} \cosh(2k-1)\pi = 2 \int_0^1 1 \cos \frac{(2k-1)\pi}{2} y dy$$

$$= \frac{4}{(2k-1)\pi} \sin \frac{(2k-1)\pi}{2} = \frac{4(-1)^{k-1}}{(2k-1)\pi}.$$

$$b_k = \frac{B_k}{\frac{(2k-1)\pi}{2} \cosh(2k-1)\pi} = \frac{8(-1)^{k-1}}{(2k-1)^2 \pi^2 \cosh(2k-1)\pi}.$$

$$u(x, y) = \sum_{k=1}^{\infty} \frac{8(-1)^{k-1}}{(2k-1)^2 \pi^2 \cosh(2k-1)\pi} \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y.$$

(2) Riešte okrajovú úlohu

$$\Delta u(x, y) = 1, \quad x^2 + y^2 < 1, \quad u(x, y) = 0, \quad \text{ak } x^2 + y^2 = 1.$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 1, \quad 0 < r < 1.$$

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = r^2, \quad 0 \leq r < 1, \quad 0 \leq \phi \leq 2\pi, \quad u(1, \phi) = 0$$

$$u(r, \phi) = R(r), \quad 0 \leq r < 1, \quad R(1) = 0.$$

$$r^2 R''(r) + r R'(r) = r^2, \quad r R'' + R' = r,$$

$$(r R'(r))' = r, \quad R(1) = 0,$$

$$r R'(r) = \frac{1}{2} r^2 + C, \quad R'(r) = \frac{1}{2} r + \frac{C_1}{r},$$

$$R(r) = \frac{1}{4} r^2 + C_1 \ln r + C_2, \quad C_1 = 0, \quad \text{pretože riešenie je ohraničené.}$$

$$R(1) = \frac{1}{4} + C_2 = 0 \Rightarrow C_2 = -\frac{1}{4},$$

$$R(r) = u(r, \phi) = \frac{1}{4} r^2 - \frac{1}{4} = \frac{1}{4} (r^2 - 1),$$

$$u(x, y) = \frac{1}{4} (x^2 + y^2 - 1).$$

(3) Riešte okrajovú úlohu

$$\Delta u(x, y) = 0, \quad x^2 + y^2 < 9, \quad y > 0;$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial n} = 2, \quad \text{ak } x^2 + y^2 = 9.$$

Riešenie:

$$\Delta u(r, \varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad u(r, 0) = u(r, \pi) = 0, \quad \frac{\partial u}{\partial r}(3, \varphi) = 2.$$

$$r^2 \Delta u(r, \varphi) = r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < 3, \quad 0 < \varphi < \pi.$$

Dosadiť $u(r, \varphi) = R(r)\Phi(\varphi)$:

$$r^2 R''\Phi + rR'\Phi + R\Phi'' = 0 / \frac{1}{R\Phi},$$

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

$$r^2 R'' + rR' - \lambda R(r) = 0, \quad 0 \leq r < 3,$$

$$\Phi'' + \lambda \Phi = 0, \quad \Phi(0) = \Phi(\pi) = 0.$$

Z prvej okrajovej podmienky: $\lambda > 0$, $\Phi(\varphi) = \sin \sqrt{\lambda} \varphi$.

Z druhej okrajovej podmienky:

$$\Phi(\pi) = \sin \sqrt{\lambda} \pi = 0, \Rightarrow \sqrt{\lambda} \pi = n\pi, \quad n = 1, 2, \dots$$

$$\lambda \equiv \lambda_n = n^2, \quad \Phi \equiv \Phi_n(\varphi) = \sin n\varphi.$$

$$r^2 R'' + rR' - n^2 R_n(r) = 0, \quad 0 \leq r < 3,$$

$$R_n(r) = a_n r^n + b_n r^{-n}, \quad b_n = 0 \text{ - ohraničenosť riešenia v nule.}$$

$$u_n(r, \varphi) = R_n(r)\Phi_n(\varphi) = a_n r^n \sin n\varphi.$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} a_n r^n \sin n\varphi.$$

Nenulová okrajová podmienka:

$$\frac{\partial u}{\partial r}(3, \varphi) = 2 = \sum_{n=1}^{\infty} a_n n 3^{n-1} \sin n\varphi.$$

Fourierove koeficienty:

$$A_n = a_n n 3^{n-1} = \frac{2}{\pi} \int_0^\pi 2 \sin n\varphi d\varphi = \frac{2[1 - (-1)^n]}{n\pi}.$$

$$a_n = \frac{2[1 - (-1)^n]}{3^{n-1} n^2 \pi}.$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{6[1 - (-1)^n]}{n^2} \left(\frac{r}{3}\right)^n \sin n\varphi.$$

(4) Riešte okrajovú úlohu

$$\Delta u(x, y) = 0, \quad x^2 + y^2 < 4, \quad x > 0, \quad y > 0;$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial x}(0, y) = 0, \quad u = 1, \quad \text{ak } x^2 + y^2 = 4.$$

Riešenie:

$$\Delta u(r, \varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad u(r, 0) = u(r, \pi) = 0, \quad u(2, \varphi) = 1.$$

$$r^2 \Delta u(r, \varphi) = r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < 2, \quad 0 < \varphi < \frac{\pi}{2}.$$

Dosadiť $u(r, \varphi) = R(r)\Phi(\varphi)$:

$$r^2 R''\Phi + rR'\Phi + R\Phi'' = 0 / \frac{1}{R\Phi},$$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

$$r^2 R'' + r R' - \lambda R(r) = 0, \quad 0 \leq r < 2,$$

$$\Phi'' + \lambda \Phi = 0, \quad \Phi(0) = \Phi'(\frac{\pi}{2}) = 0.$$

Z prvej okrajovej podmienky: $\lambda > 0$, $\Phi(\varphi) = \sin \sqrt{\lambda} \varphi$.

Z druhej okrajovej podmienky:

$$\Phi'(\frac{\pi}{2}) = \sqrt{\lambda} \cos \sqrt{\lambda} \frac{\pi}{2} = 0, \Rightarrow \sqrt{\lambda} \frac{\pi}{2} = (2n-1) \frac{\pi}{2}, \quad n = 1, 2, \dots$$

$$\lambda \equiv \lambda_n = (2n-1)^2, \quad \Phi \equiv \Phi_n(\varphi) = \sin(2n-1)\varphi.$$

$$r^2 R_n'' + r R_n' - (2n-1)^2 R_n(r) = 0, \quad 0 \leq r < 2,$$

$$R_n(r) = a_n r^{2n-1} + b_n r^{-(2n-1)}, \quad b_n = 0 \text{ - ohraničenosť riešenia v nule.}$$

$$u_n(r, \varphi) = R_n(r) \Phi_n(\varphi) = a_n r^{2n-1} \sin(2n-1)\varphi.$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} a_n r^{2n-1} \sin(2n-1)\varphi.$$

Nenulová okrajová podmienka:

$$u(2, \varphi) = 1 = \sum_{n=1}^{\infty} a_n 2^{2n-1} \sin(2n-1)\varphi.$$

Fourierove koeficienty:

$$A_n = a_n 2^{2n-1} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin(2n-1)\varphi \, d\varphi = \frac{4}{(2n-1)\pi}.$$

$$a_n = \frac{4}{2^{2n-1}(2n-1)\pi}.$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \left(\frac{r}{2}\right)^{2n-1} \sin(2n-1)\varphi.$$

(5) Riešte úlohu na vlastné hodnoty a vlastné funkcie

$$\Delta u + \lambda u(x, y) = 0, \quad 0 < x < \pi, \quad 0 < y < 2,$$

$$u(0, y) = u(\pi, y) = \frac{\partial u}{\partial y}(x, 0) = u(x, 2) = 0.$$

Riešenie:

$$u(x, y) = X(x)Y(y), \quad \lambda = \mu + \nu.$$

Jednorozmerné úlohy na vlastné hodnoty a vlastné funkcie:

$$\text{a) } X'' + \mu X(x) = 0, \quad X(0) = X(\pi) = 0,$$

$$\text{b) } Y'' + \nu Y(y) = 0, \quad Y'(0) = Y(2) = 0.$$

$$\text{a): Z prvej okrajovej podmienky: } \mu > 0, \quad X(x) = \sin \sqrt{\mu} x,$$

Z druhej okrajovej podmienky:

$$X(\pi) = \sin(\sqrt{\mu}\pi) = 0 \Rightarrow \sqrt{\mu}\pi = m\pi.$$

$$\mu \equiv \mu_m = m^2, \quad X \equiv X_m(x) = \sin mx.$$

b): Z druhej okrajovej podmienky: $\nu > 0$,
 Z prvej okrajovej podmienky: $Y(y) = \cos \sqrt{\nu}y$,

Z druhej okrajovej podmienky:
 $Y(2) = \cos \sqrt{\nu}2 = 0 \Rightarrow \sqrt{\nu}2 = (2n-1)\frac{\pi}{2}$.

$$\nu \equiv \nu_n = \left[\frac{(2n-1)\pi}{4} \right]^2, \quad Y \equiv Y_n(y) = \cos \frac{(2n-1)\pi}{4}y.$$

Spolu:

$$\lambda \equiv \lambda_{m,n} = \mu_m + \nu_n = m^2 + \left[\frac{(2n-1)\pi}{4} \right]^2,$$

$$u \equiv u_{m,n}(x,y) = X_m(x)Y_n(y) = \sin mx \cos \frac{(2n-1)\pi}{4}y, \quad m, n = 1, 2, \dots$$

(6) Riešte úlohu na vlastné hodnoty a vlastné funkcie

$$\Delta u + \lambda u(x,y) = 0, \quad 0 < x < 1, \quad 0 < y < 2, \\ \frac{\partial u}{\partial x} u(0,y) = u(1,y) = u(x,0) = u(x,2) = 0.$$

Riešenie:

$$u(x,y) = X(x)Y(y), \quad \lambda = \mu + \nu.$$

Jednorozmerné úlohy na vlastné hodnoty a vlastné funkcie:

$$a) \quad X'' + \mu X(x) = 0, \quad X'(0) = X(1) = 0,$$

$$b) \quad Y'' + \nu Y(y) = 0, \quad Y(0) = Y(2) = 0.$$

a): Z druhej okrajovej podmienky: $\mu > 0$,
 Z prvej okrajovej podmienky: $X(x) = \cos \sqrt{\mu}x$,
 Z druhej okrajovej podmienky:
 $X(1) = \cos(\sqrt{\mu}) = 0 \Rightarrow \sqrt{\mu} = (2m-1)\frac{\pi}{2}$.

$$\mu \equiv \mu_m = \left[\frac{(2m-1)\pi}{2} \right]^2, \quad X \equiv X_m(x) = \cos \frac{(2m-1)\pi}{2}x.$$

$$b): \text{ Z prvej okrajovej podmienky: } \nu > 0, \quad Y(y) = \sin \sqrt{\nu}y,$$

Z druhej okrajovej podmienky:
 $Y(2) = \sin \sqrt{\nu}2 = 0 \Rightarrow \sqrt{\nu}2 = n\pi$.

$$\nu \equiv \nu_n = \left(\frac{n\pi}{2} \right)^2, \quad Y \equiv Y_n(y) = \sin \frac{n\pi}{2}y.$$

Spolu:

$$\lambda \equiv \lambda_{m,n} = \mu_m + \nu_n = \left[\frac{(2m-1)\pi}{2} \right]^2 + \left(\frac{n\pi}{2} \right)^2 = \left(\frac{\pi}{2} \right)^2 [(2m-1)^2 + n^2],$$

$$u \equiv u_{m,n}(x,y) = X_m(x)Y_n(y) = \cos \frac{(2m-1)\pi}{2}x \sin \frac{n\pi}{2}y, \quad m, n = 1, 2, \dots$$