Quasi-Static Hemivariational Inequalities with Applications to Frictional Viscoelastic Contact Problems

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Orientation: two classes of operator inclusions

We consider subdifferential inclusions

- history–dependent subdifferential inclusions

$$\boldsymbol{A}(\boldsymbol{t}, \boldsymbol{u}(\boldsymbol{t})) + \boldsymbol{\mathcal{S}}\boldsymbol{u}(\boldsymbol{t}) + \gamma^* \,\boldsymbol{\partial}\boldsymbol{J}(\boldsymbol{t}, \gamma \boldsymbol{u}(\boldsymbol{t})) \ni \boldsymbol{f}(\boldsymbol{t}) \ \text{ a.e. } \boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{T})$$

- quasi–static subdifferential inclusions

$$\begin{cases} \boldsymbol{A}(\boldsymbol{t})\boldsymbol{u}'(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) + \gamma^* \,\boldsymbol{\partial}\boldsymbol{J}(\boldsymbol{t}, \gamma \boldsymbol{u}(\boldsymbol{t})) \ni \boldsymbol{f}(\boldsymbol{t}) \ \text{ a.e. } \boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{T}) \\ \boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{u}_{\boldsymbol{0}} \end{cases}$$

and the corresponding two classes of hemivariational inequalities.

First class of problems: outline of the talk

- A class of history–dependent subdifferential inclusions
- A class of history–dependent hemivariational inequalities
- Application to a frictional quasistatic contact problem

History–dependent subdifferential inclusions

- A class of history–dependent subdifferential inclusions
  - Hemicontinuous and strongly monotone operator
  - History dependent nonlinear operator
  - Surjectivity result for pseudomonotone and coercive operator
  - Fixed point argument
- A class of history–dependent hemivariational inequalities
  - Nonconvex and locally Lipschitz potential on the contact boundary
  - Existence result
  - Uniqueness of solutions under Clarke regularity hypothesis
- A frictional quasistatic contact problem
  - Two nonconvex potentials in subdifferential boundary conditions
  - Nonlinear strongly monotone viscosity operator
  - Lipschitz continuous elasticity operator
  - Hemivariational inequality formulation in velocity
  - Main results on existence, uniqueness and regularity of weak solutions

History–dependent subdifferential inclusions

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset with a Lipschitz boundary $\partial\Omega$ and $\Gamma \subseteq \partial\Omega$,

$$V \subset H^1(\Omega;\mathbb{R}^s) \text{ a closed subspace}, \quad s \geq 1,$$

$$H = L^2(\Omega;\mathbb{R}^s) \ \text{ and } \ Z = H^\delta(\Omega;\mathbb{R}^s), \quad \delta \in (1/2, 1).$$

Denoting by $i\colon V \longrightarrow Z$ the embedding, by $\gamma\colon Z \longrightarrow L^2(\Gamma;\mathbb{R}^s)$ and $\gamma_0\colon H^1(\Omega;\mathbb{R}^s) \rightarrow H^{1/2}(\Gamma;\mathbb{R}^s) \subset L^2(\Gamma;\mathbb{R}^s)$ the trace operators, we have $\gamma_0 v = \gamma(iv)$ for all $v \in V$. For simplicity, we write $\gamma_0 v = \gamma v$ for all $v \in V$.

From the theory of Sobolev spaces, we know that

$$(V, H, V^*) \ \text{ and } \ (Z, H, Z^*)$$

form **evolution triples** of spaces and $V \subset Z$ is compact.

Denote by c_e the embedding constant of V into Z.

History–dependent subdifferential inclusions

We also introduce the following spaces

$$\mathcal{V} = L^2(0,T;V), \;\; \mathcal{Z} = L^2(0,T;Z) \;\text{ and }\; \widehat{\mathcal{H}} = L^2(0,T;H),$$

where $0 < T < +\infty$. Then

$$\mathcal{V} \subseteq \mathcal{Z} \subseteq \widehat{\mathcal{H}} \subseteq \mathcal{Z}^* \subseteq \mathcal{V}^*$$

are continuous with $\mathcal{Z}^* = L^2(0,T;Z^*)$ and $\mathcal{V}^* = L^2(0,T;V^*)$.

Let $A\colon (0,T)\times V \to V^*$, $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$, $J\colon (0,T)\times L^2(\Gamma;\mathbb{R}^s) \to \mathbb{R}$ and $f\colon (0,T) \to V^*$ be given.

PROBLEM 1 *Find* $u \in \mathcal{V}$ *such that*

$$A(t,u(t)) + \mathcal{S}u(t) + \gamma^*\,\partial J(t,\gamma u(t)) \ni f(t) \tag{1}$$

for a.e. $t \in (0,T)$.

Subdifferential of a locally Lipschitz function

Given a locally Lipschitz function $h\colon E \to \mathbb{R}$, where E is a Banach space, we define (Clarke (1983)):

- **the generalized directional derivative** of h at $x \in E$ in the direction $v \in E$ by

$$h^0(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{h(y+tv)-h(y)}{t}.$$

- **the generalized gradient** of h at x by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* : h^0(x;v) \geq \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\}.$$

The locally Lipschitz function h is called **regular** (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in E$.

History–dependent subdifferential inclusions

To avoid any confusion, we note that the notation $\mathcal{S}u(t)$ stands for $(\mathcal{S}u)(t)$, i.e. $\mathcal{S}u(t) = (\mathcal{S}u)(t)$ for all $u \in \mathcal{V}$ and a.e. $t \in (0, T)$.

The symbol ∂J denotes the Clarke subdifferential of a locally Lipschitz function $J(t, \cdot)$.

DEFINITION 2 *A function $u \in \mathcal{V}$ is called* **a solution** *to Problem* 1 *if and only if there exists $\zeta \in \mathcal{Z}^*$ such that*

$$\begin{cases} A(t, u(t)) + \mathcal{S}u(t) + \zeta(t) = f(t) \ \text{ a.e. } t \in (0, T), \\ \zeta(t) \in \gamma^* \partial J(t, \gamma u(t)) \ \text{ a.e. } t \in (0, T). \end{cases}$$

History–dependent subdifferential inclusions

We need the following hypothesis $H(A)$:

$A\colon (0,T)\times V \to V^*$ is such that

(a) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;

(b) $A(t,\cdot)$ is hemicontinuous and strongly monotone for a.e. $t \in (0,T)$, i.e. $\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{V^*\times V} \geq$ $\geq m_1 \|v_1 - v_2\|_V^2$ for all $v_1, v_2 \in V$ with $m_1 > 0$;

(c) $\|A(t,v)\|_{V^*} \leq a_0(t) + a_1\|v\|_V$ for all $v \in V$, a.e. $t \in (0,T)$ with $a_0 \in L^2(0,T), a_0 \geq 0$ and $a_1 > 0$;

(d) $A(t,0) = 0$ for a.e. $t \in (0,T)$.

History–dependent subdifferential inclusions

We need the following hypothesis $H(\mathcal{S})$:

$$\left.\begin{array}{l} \mathcal{S} : \mathcal{V} \to \mathcal{V}^* \text{ is such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_{V^*} \le L_{\mathcal{S}} \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_V \, ds \\ \quad \text{for all } u_1,\ u_2 \in \mathcal{V},\ \text{a.e. } t \in (0,T) \text{ with } L_{\mathcal{S}} > 0. \end{array}\right\}$$

History–dependent subdifferential inclusions

The hypothesis $H(\mathcal{S})$ is satisfied for $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ given by

$$\mathcal{S}v(t) = R\Big(t, \int_0^t v(s)\, ds + v_0\Big) \quad \text{for all } v \in \mathcal{V}, \text{ a.e. } t \in (0,T),$$

where $R\colon (0,T) \times V \to V^*$ is such that $R(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$, $R(t,\cdot)$ is a Lipschitz continuous operator for a.e. $t \in (0,T)$ and $v_0 \in V$.

It is also satisfied for the **Volterra operator** $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ given by

$$\mathcal{S}v(t) = \int_0^t C(t-s)\, v(s)\, ds \quad \text{for all } v \in \mathcal{V}, \text{ a.e. } t \in (0,T),$$

where $C \in L^\infty(0,T;\mathcal{L}(V,V^*))$.

In the cases above the current value $\mathcal{S}v(t)$ at the moment t depends on the history of the values of v at the moments $0 \le s \le t$ and, therefore, we refer the above operators as *history-dependent operators*.

History–dependent subdifferential inclusions

We need the following hypothesis $H(J)$:

$J\colon (0,T)\times L^2(\Gamma;\mathbb{R}^s)\longrightarrow \mathbb{R}$ is such that

(a) $J(\cdot,u)$ is measurable on $(0,T)$ for all $u\in L^2(\Gamma;\mathbb{R}^s)$;

(b) $J(t,\cdot)$ is locally Lipschitz on $L^2(\Gamma;\mathbb{R}^s)$ for a.e. $t\in(0,T)$;

(c) $\|\partial J(t,u)\|_{L^2(\Gamma;\mathbb{R}^s)}\le c_0+c_1\,\|u\|_{L^2(\Gamma;\mathbb{R}^s)}$ for all $u\in L^2(\Gamma;\mathbb{R}^s)$, a.e. $t\in(0,T)$ with $c_0,c_1\ge 0$;

(d) $(z_1-z_2,u_1-u_2)_{L^2(\Gamma;\mathbb{R}^s)}\ge -m_2\|u_1-u_2\|^2_{L^2(\Gamma;\mathbb{R}^s)}$ for all $z_i\in\partial J(t,u_i)$, $u_i,z_i\in L^2(\Gamma;\mathbb{R}^s)$, $i=1,2$, a.e. $t\in(0,T)$ with $m_2\ge 0$;

(e) $J^0(t,u;-u)\le d_0\left(1+\|u\|_{L^2(\Gamma;\mathbb{R}^s)}\right)$ for all $u\in L^2(\Gamma;\mathbb{R}^s)$, a.e. $t\in(0,T)$ with $d_0\ge 0$.

History–dependent subdifferential inclusions

LEMMA 3 Assume $H(A)$ and $f \in \mathcal{V}^*$. If one of the following hypotheses

(i) $H(J)$(a)–(d) and $m_1 > \max\{c_1, m_2\}\, c_e^2\, \|\gamma\|^2$

(ii) $H(J)$ and $m_1 > m_2\, c_e^2\, \|\gamma\|^2$

is satisfied, then the problem

$$A(t, u(t)) + \gamma^* \partial J(t, \gamma u(t)) \ni f(t) \ \text{ a.e. } t \in (0, T)$$

has a unique solution $u \in \mathcal{V}$.

THEOREM 4 Assume $H(A)$, $H(\mathcal{S})$ and $f \in \mathcal{V}^*$. If either (i) or (ii) of the hypothesis of Lemma 3 holds, then Problem 1

$$A(t, u(t)) + \mathcal{S}u(t) + \gamma^* \partial J(t, \gamma u(t)) \ni f(t) \ \text{ a.e. } t \in (0, T)$$

has a unique solution.

History–dependent hemivariational inequalities

PROBLEM 5 *Find* $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$ *such that*

$$\begin{aligned}\langle \boldsymbol{A}(\boldsymbol{t}, \boldsymbol{u}(\boldsymbol{t})), \boldsymbol{v}\rangle_{V^*\times V} \; &+ \; \langle \boldsymbol{\mathcal{S}u}(\boldsymbol{t}), \boldsymbol{v}\rangle_{V^*\times V} + \\ &+ \; \int_{\Gamma} \boldsymbol{j}^{0}(\boldsymbol{t}, \gamma \boldsymbol{u}(\boldsymbol{t}); \gamma \boldsymbol{v})\, \boldsymbol{d\Gamma} \geq \langle \boldsymbol{f}(\boldsymbol{t}), \boldsymbol{v}\rangle_{V^*\times V}\end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ *and a.e.* $\boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{T})$.

History–dependent hemivariational inequalities

We need the following hypotheses $H(j)$:

$$\left.\begin{array}{l}
j\colon \Gamma \times (0,T) \times \mathbb{R}^s \longrightarrow \mathbb{R} \text{ is such that} \\
\quad \text{(a) } j(\cdot,\cdot,\xi) \text{ is measurable on } \Gamma \times (0,T) \text{ for all } \xi \in \mathbb{R}^s \text{ and} \\
\qquad j(\cdot,\cdot,0) \in L^1(\Gamma \times (0,T)); \\
\quad \text{(b) } j(x,t,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^s \text{ for a.e.} \\
\qquad (x,t) \in \Gamma \times (0,T); \\
\quad \text{(c) } \|\partial j(x,t,\xi)\|_{\mathbb{R}^s} \le \overline{c}_0 + \overline{c}_1 \|\xi\|_{\mathbb{R}^s} \text{ for a.e. } (x,t) \in \Gamma \times (0,T), \\
\qquad \text{all } \xi \in \mathbb{R}^s \text{ with } \overline{c}_0, \overline{c}_1 \ge 0; \\
\quad \text{(d) } (\zeta_1 - \zeta_2)\cdot(\xi_1 - \xi_2) \ge -m_2 \|\xi_1 - \xi_2\|^2_{\mathbb{R}^s} \text{ for all } \zeta_i, \xi_i \in \mathbb{R}^s, \\
\qquad \zeta_i \in \partial j(x,t,\xi_i),\, i = 1,2, \text{ a.e. } (x,t) \in \Gamma \times (0,T) \\
\qquad \text{with } m_2 \ge 0; \\
\quad \text{(e) } j^0(x,t,\xi;-\xi) \le \overline{d}_0\,(1 + \|\xi\|_{\mathbb{R}^s}) \text{ for a.e. } (x,t) \in \Gamma \times (0,T), \\
\qquad \text{all } \xi \in \mathbb{R}^s \text{ with } \overline{d}_0 \ge 0.
\end{array}\right\}$$

History–dependent hemivariational inequalities

THEOREM 6 Assume that $\boldsymbol{H(A)}$ and $\boldsymbol{H(\mathcal{S})}$ hold and $\boldsymbol{f \in \mathcal{V}^*}$. If one of the following hypotheses

i) $\boldsymbol{H(j)}$(a)–(d) and $\boldsymbol{m_1 > \max\{\sqrt{3}\,\bar{c}_1, m_2\}\, c_e^2}\, \|\gamma\|^2$

ii) $\boldsymbol{H(j)}$ and $\boldsymbol{m_1 > m_2\, c_e^2}\, \|\gamma\|^2$

is satisfied, then the hemivariational inequality in Problem 5 has a solution $\boldsymbol{u \in \mathcal{V}}$. If, in addition, the regularity condition

either $\boldsymbol{j(x, t, \cdot)}$ or $\boldsymbol{-j(x, t, \cdot)}$ is regular on $\mathbb{R}^s$ for a.e. $\boldsymbol{(x, t) \in \Gamma \times (0, T)}$

holds, then the solution of Problem 5 is unique.

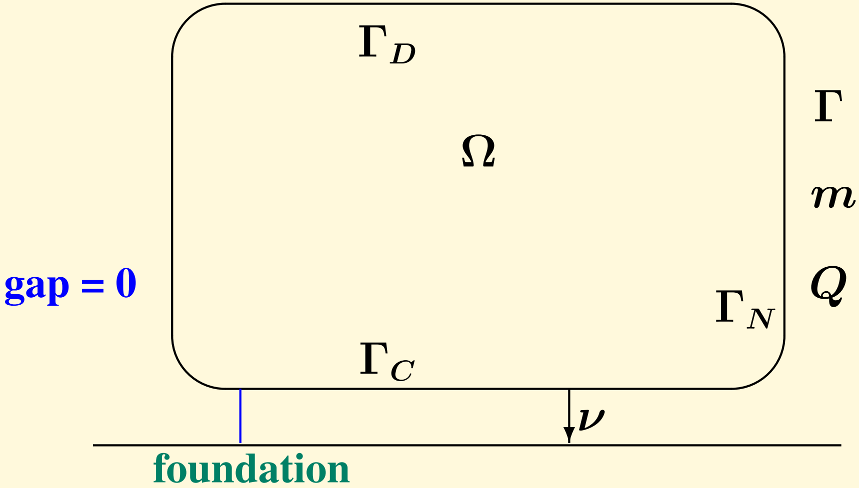
A frictional contact problem – classical formulation

We illustrate the use of a result for time-dependent hemivariational inequality in the study of **a quasistatic contact problem** in which the main variable is the velocity field. A result on the unique weak solvability of the contact problem is provided.

PROBLEM 7 *Find the displacement field* $\boldsymbol{u}\colon \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ *and the stress field* $\boldsymbol{\sigma}\colon \Omega \times [0,T] \longrightarrow \mathbb{S}^d$ *such that, for all* $t \in (0,T)$,

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) && \text{in } \Omega \\
\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) &= \boldsymbol{0} && \text{in } \Omega \\
\boldsymbol{u}(t) &= \boldsymbol{0} && \text{on } \Gamma_D \\
\boldsymbol{\sigma}(t)\boldsymbol{\nu} &= \boldsymbol{f}_N(t) && \text{on } \Gamma_N \\
-\sigma_\nu(t) &\in \partial j_\nu(t, u'_\nu(t)) && \text{on } \Gamma_C \\
-\boldsymbol{\sigma}_\tau(t) &\in \partial j_\tau(t, \boldsymbol{u}'_\tau(t)) && \text{on } \Gamma_C \\
\boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega.
\end{aligned}
$$

A frictional contact problem – physical setting



$\Gamma = \partial\Omega$

$m(\Gamma_D) > 0$

$Q = \Omega \times (0, T)$

$\Omega \subset \mathbb{R}^d$ is occupied by a viscoelastic body
$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ mutually disjoint measurable parts
Γ_C the contact surface

We suppose that the body is clamped on Γ_D and so the displacement field vanishes there. Surface tractions of density f_N act on Γ_N and volume forces of density f_0 act in Ω. We assume that the forces and tractions change slowly in time so that the acceleration of the system is negligible and, therefore, the process is quasistatic.

A frictional contact problem – notation

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor. Let $\boldsymbol{Q} = \boldsymbol{\Omega} \times (0, \boldsymbol{T})$, $\boldsymbol{\Sigma}_D = \boldsymbol{\Gamma}_D \times (0, \boldsymbol{T})$, $\boldsymbol{\Sigma}_N = \boldsymbol{\Gamma}_N \times (0, \boldsymbol{T})$ and $\boldsymbol{\Sigma}_C = \boldsymbol{\Gamma}_C \times (0, \boldsymbol{T})$,

The normal and tangential components of the displacement and stress are denoted by

$$\boldsymbol{v}_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - \boldsymbol{v}_\nu \boldsymbol{\nu},$$

$$\boldsymbol{\sigma}_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{\sigma}_\nu \boldsymbol{\nu}.$$

We introduce the spaces

$$\boldsymbol{V} = \{\, \boldsymbol{v} \in \boldsymbol{H}^1(\boldsymbol{\Omega}; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \boldsymbol{\Gamma}_D \,\}, \quad \boldsymbol{\mathcal{H}} = \boldsymbol{L}^2(\boldsymbol{\Omega}; \mathbb{S}^d).$$

The space $\boldsymbol{\mathcal{H}}$ is a real Hilbert space with the inner product given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \boldsymbol{\sigma}_{ij}(\boldsymbol{x}) \, : \, \boldsymbol{\tau}_{ij}(\boldsymbol{x}) \, d\boldsymbol{x} \;\text{ for all }\; \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}.$$

Since $\text{meas}(\boldsymbol{\Gamma}_D) > \boldsymbol{0}$, it is well known that $\boldsymbol{V}$ is a real Hilbert space with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$.

A frictional contact problem

We assume that the viscosity operator $\mathcal{A}$ satisfies $H(\mathcal{A})$:

$$\left.\begin{array}{l}
\mathcal{A}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\
\quad \text{(a) } \mathcal{A}(\cdot, \cdot, \varepsilon) \text{ is measurable on } Q \text{ for all } \varepsilon \in \mathbb{S}^d; \\
\quad \text{(b) } \mathcal{A}(x, t, \cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } (x, t) \in Q; \\
\quad \text{(c) } (\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d} \\
\qquad\quad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } (x, t) \in Q \text{ with } m_{\mathcal{A}} > 0; \\
\quad \text{(d) } \|\mathcal{A}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq \overline{a}_0(x, t) + \overline{a}_1 \|\varepsilon\|_{\mathbb{S}^d} \text{ for all } \varepsilon \in \mathbb{S}^d, \\
\qquad\quad \text{a.e. } (x, t) \in Q \text{ with } \overline{a}_0 \in L^2(Q), \overline{a}_0 \geq 0 \text{ and } \overline{a}_1 > 0; \\
\quad \text{(e) } \mathcal{A}(x, t, 0) = 0 \text{ for a.e. } (x, t) \in Q.
\end{array}\right\}$$

A frictional contact problem

We assume that the elasticity operator $\mathcal{B}$ satisfies $H(\mathcal{B})$:

$$\left.\begin{array}{l} \mathcal{B}\colon Q \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ is such that} \\ \quad \text{(a) } \mathcal{B}(\cdot,\cdot,\varepsilon) \text{ is measurable on } Q \text{ for all } \varepsilon \in \mathbb{S}^d; \\ \quad \text{(b) } \|\mathcal{B}(x,t,\varepsilon_1) - \mathcal{B}(x,t,\varepsilon_2)\|_{\mathbb{S}^d} \le L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \\ \qquad\qquad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } (x,t) \in Q \text{ with } L_{\mathcal{B}} > 0; \\ \quad \text{(c) } \mathcal{B}(\cdot,\cdot,0) \in L^2(Q;\mathbb{S}^d). \end{array}\right\}$$

A frictional contact problem

We assume that the contact potential j_ν satisfies $H(j_\nu)$:

$$\left.\begin{array}{l} j_\nu\colon \Sigma_C \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\nu(\cdot,\cdot,r) \text{ is measurable on } \Sigma_C \text{ for all } r \in \mathbb{R} \text{ and} \\ \qquad\quad j_\nu(\cdot,\cdot,0) \in L^1(\Sigma_C); \\ \quad \text{(b) } j_\nu(x,t,\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } (x,t) \in \Sigma_C; \\ \quad \text{(c) } |\partial j_\nu(x,t,r)| \le c_{0\nu} + c_{1\nu}|r| \text{ for all } r \in \mathbb{R}, \text{ a.e. } (x,t) \in \Sigma_C \\ \qquad\quad \text{with } c_{0\nu}, c_{1\nu} \ge 0; \\ \quad \text{(d) } (\zeta_1 - \zeta_2)(r_1 - r_2) \ge -m_\nu |r_1 - r_2|^2 \text{ for all } \zeta_i \in \partial j_\nu(x,t,r_i), \\ \qquad\quad r_i \in \mathbb{R}, i = 1,2, \text{ a.e.} (x,t) \in \Sigma_C \text{ with } m_\nu \ge 0; \\ \quad \text{(e) } j_\nu^0(x,t,r;-r) \le d_\nu(1 + |r|) \text{ for all } r \in \mathbb{R}, \text{ a.e. } (x,t) \in \Sigma_C \\ \qquad\quad \text{with } d_\nu \ge 0. \end{array}\right\}$$

A frictional contact problem

We assume that the contact potential j_τ satisfies $H(j_\tau)$:

$$\left.\begin{array}{l}
j_\tau\colon \Sigma_C \times \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is such that} \\
\quad \text{(a) } j_\tau(\cdot,\cdot,\xi) \text{ is measurable on } \Sigma_C \text{ for all } \xi \in \mathbb{R}^d \text{ and} \\
\qquad\quad j_\tau(\cdot,\cdot,0) \in L^1(\Sigma_C); \\
\quad \text{(b) } j_\tau(x,t,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } (x,t) \in \Sigma_C; \\
\quad \text{(c) } \|\partial j_\tau(x,t,\xi)\|_{\mathbb{R}^d} \le c_{0\tau} + c_{1\tau}\|\xi\|_{\mathbb{R}^d} \text{ for all } \xi \in \mathbb{R}^d, \\
\qquad\quad \text{a.e. } (x,t) \in \Sigma_C \text{ with } c_{0\tau}, c_{1\tau} \ge 0; \\
\quad \text{(d) } (\zeta_1 - \zeta_2)\cdot(\xi_1 - \xi_2) \ge -m_\tau\|\xi_1 - \xi_2\|^2_{\mathbb{R}^d} \text{ for all} \\
\qquad\quad \zeta_i \in \partial j_\tau(x,t,\xi_i), \xi_i \in \mathbb{R}^d, i = 1,2, \text{ a.e. } (x,t) \in \Sigma_C \\
\qquad\quad \text{with } m_\tau \ge 0; \\
\quad \text{(e) } j_\tau^0(x,t,\xi;-\xi) \le d_\tau\left(1 + \|\xi\|_{\mathbb{R}^d}\right) \text{ for all } \xi \in \mathbb{R}^d, \\
\qquad\quad \text{a.e. } (x,t) \in \Sigma_C \text{ with } d_\tau \ge 0.
\end{array}\right\}$$

A frictional contact problem

We assume that the volume force and traction densities, and the initial displacement satisfy (H_0):

$$f_0 \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \qquad f_N \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)),$$

$$u_0 \in V.$$

A frictional contact problem

Introducing the function $f\colon (0,T) \longrightarrow V^*$ given by

$$\langle f(t), v\rangle_{V^*\times V} = (f_0(t), v)_{L^2(\Omega;\mathbb{R}^d)} + (f_N(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)}$$

for all $v \in V$ and a.e. $t \in (0,T)$, we are lead to the following variational formulation of Problem 7:

$$(\mathcal{A}(t, \varepsilon(u'(t))), \varepsilon(v))_{\mathcal{H}} + (\mathcal{B}(t, \varepsilon(u(t))), \varepsilon(v))_{\mathcal{H}} + \\ + \int_{\Gamma_C} \left(j_\nu^0(t, u_\nu'(t); v_\nu) + j_\tau^0(t, u_\tau'(t); v_\tau)\right)\, d\Gamma \geq \langle f(t), v\rangle_{V^*\times V}$$

for all $v \in V$ and a.e. $t \in (0,T)$.

A frictional contact problem

Let $w = u'$ denote the velocity field. Then, by using the initial condition, it follows that

$$u(t) = \int_0^t w(s)\, ds + u_0 \ \text{ for all } \ t \in [0, T].$$

Therefore, the weak formulation can be formulated in terms of velocity:

PROBLEM 8 *Find a velocity field* $w \in \mathcal{V}$ *such that*

$$(\mathcal{A}(t, \varepsilon(w(t))), \varepsilon(v))_{\mathcal{H}} + \Big(\mathcal{B}\big(t, \varepsilon(\int_0^t w(s)\, ds + u_0)\big), \varepsilon(v)\Big)_{\mathcal{H}} +$$

$$+ \int_{\Gamma_C} \left(j_\nu^0(t, w_\nu(t); v_\nu) + j_\tau^0(t, w_\tau(t); v_\tau)\right)\, d\Gamma \geq \langle f(t), v\rangle_{V^* \times V}$$

for all $v \in V$ *and a.e.* $t \in (0, T)$.

A frictional contact problem

Our main result in the study of Problem 8 is the following.

THEOREM 9 Assume that $H(\mathcal{A})$, $H(\mathcal{B})$ and (H_0) hold. If one of the following hypotheses

i) $H(j_\nu)$(a)–(d), $H(j_\tau)$(a)–(d) and

$$m_{\mathcal{A}} > \max\left\{\sqrt{3}(c_{1\nu} + c_{1\tau}), m_\nu, m_\tau\right\} c_e^2 \, \|\gamma\|^2$$

ii) $H(j_\nu)$, $H(j_\tau)$ and $m_{\mathcal{A}} > \max\{m_\nu, m_\tau\} \, c_e^2 \, \|\gamma\|^2$

is satisfied, then Problem 8 has at least one solution. If, in addition,

$$\left.\begin{array}{l} \text{either } j_\nu(x,t,\cdot) \text{ or } -j_\nu(x,t,\cdot) \text{ is regular on } \mathbb{R} \text{ and} \\ \text{either } j_\tau(x,t,\cdot) \text{ or } -j_\tau(x,t,\cdot) \text{ is regular on } \mathbb{R}^d \\ \text{for a.e. } (x,t) \in \Sigma_C, \end{array}\right\}$$

then the solution of Problem 8 is unique.

A frictional contact problem – conclusion

Let $\boldsymbol{w}$ be a solution of Problem 8 and let $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ be the functions defined by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0 \text{ and } \boldsymbol{\sigma}(t) = \mathcal{A}(t, \varepsilon(\boldsymbol{u}'(t))) + \mathcal{B}(t, \varepsilon(\boldsymbol{u}(t))).$$

Then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ is called **a weak solution** of the quasistatic frictional contact problem.

We conclude that, under the hypotheses of Theorem 9, the quasistatic frictional contact problem has at least one weak solution with the following regularity

$$\boldsymbol{u} \in W^{1,2}(0,T;V),\ \boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}),\ \mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V^*).$$

If, in addition, the regularity condition on j_ν and j_τ holds, then the weak solution of Problem 7 is unique.

Second class of problems: outline of the talk

- Abstract evolution inclusion of second order
- Limit process via vanishing acceleration approach
- Existence of solutions to quasi-static subdifferential inclusions
- Application to quasi-static viscoelastic contact with nonmonotone normal compliance and friction

Abstract evolution inclusion of second order

Let V and Z be separable Banach spaces and let H be a separable Hilbert space. Suppose

$$V \subset Z \subset H \subset Z^* \subset V^*$$

with dense and continuous embeddings and that $V \subset Z$ compactly. Define the spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{Z} = L^2(0,T;Z)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$.

Consider the following second order evolution inclusion:

$$\begin{cases} u''(t) + A(t,u'(t)) + Bu(t) + F(t,u(t),u'(t)) \ni f(t) \ \text{a.e. } t \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

Hypotheses

$\underline{H(A)}$: $A\colon (0,T) \times V \longrightarrow V^*$ is an operator such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;

(ii) $A(t, \cdot)$ is pseudomonotone for every $t \in (0,T)$;

(iii) $||A(t,v)||_{V^*} \leq a(t) + b||v||$ with $a \in L^2(0,T)$, $a \geq 0$, $b > 0$;

(iv) $\langle A(t,v), v\rangle_{V^* \times V} \geq \alpha||v||^2$ for a.e. $t \in (0,T)$, for all $v \in V$ with $\alpha > 0$.

$\underline{H(B)}$: $B \in \mathcal{L}(V, V^*)$ is monotone (nonnegative) and symmetric.

$\underline{H(F)}$: $F\colon (0,T) \times V \times V \to 2^{Z^*}$ has nonempty, convex and closed values and

(i) $F(\cdot, u, v)$ is measurable for all $u, v \in V$;

(ii) $F(t, \cdot, \cdot)$ is usc from $V \times V$ into w-Z^* for a.e. $t \in (0,T)$, where $V \times V$ is endowed with $(Z \times Z)$-topology;

(iii) $\|F(t,u,v)\|_{Z^*} \le b_0(t) + b_1\|u\| + b_2\|v\|$ for $u, v \in V$ with $b_0 \in L^2(0,T)$, b_0, b_1, $b_2 \ge 0$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

$\underline{(H_1)}$: $\alpha > 2\sqrt{3}c_e(b_1 T + b_2)$, where $c_e > 0$ denotes the embedding constant of V into Z, i.e. $\|\cdot\|_Z \le c_e\|\cdot\|$.

THEOREM 10 *Under the hypotheses $H(A)$, $H(B)$, $H(F)$, (H_0) and (H_1) the evolution inclusion admits a solution.*

A sequence of evolution problems

We provide a corollary of Theorem 10. For $\varepsilon > 0$ we consider the problem

$$\begin{cases} \varepsilon u''(t) + A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni f(t) \text{ a.e. } t \\ u(0) = u_0, \quad \sqrt{\varepsilon}u'(0) = u_1. \end{cases}$$

Hypotheses

$\underline{H(A)}$: $A \in L^\infty(0,T;\mathcal{L}(V,V^*))$ is such that $A(t)$ is coercive with a constant $\alpha > 0$ for a.e. $t \in (0,T)$.

$\underline{H(B)}$: $B \in \mathcal{L}(V,V^*)$ is monotone (nonnegative) and symmetric.

$\underline{H(J)}$: $J\colon (0,T) \times X \longrightarrow \mathbb{R}$ is a function (with X being a Banach space) such that

(i) $J(\cdot, x)$ is measurable on $(0,T)$ for all $x \in X$;

(ii) $J(t,\cdot)$ is locally Lipschitz on X for a.e. $t \in (0,T)$;

(iii) $\|\partial J(t,x)\|_{X^*} \leq \overline{c}\,(1 + \|x\|_X)$ for all $X \in X$, a.e. $t \in (0,T)$ with $\overline{c} > 0$.

$\underline{H(M)}$: $M \in \mathcal{L}(Z,X)$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

$\underline{(H_1)}$: $\alpha > 2\,\overline{c}\,T\,c_e^2\,\|M\|\,\max\{1,\|M\|\}$, where $c_e > 0$ is an embedding constant of V into Z and $\|M\| = \|M\|_{\mathcal{L}(Z,X)}$.

Asymptotic analysis via vanishing acceleration

THEOREM 11 *If hypotheses* $\boldsymbol{H}(\boldsymbol{A})$, $\boldsymbol{H}(\boldsymbol{B})$, $\boldsymbol{H}(\boldsymbol{J})$, $\boldsymbol{H}(\boldsymbol{M})$, $(\boldsymbol{H}_0)$ *and* $(\boldsymbol{H}_1)$ *hold, then for every fixed* $\varepsilon > 0$ *the problem*

$$\begin{cases} \varepsilon u''(t) + A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni f(t) \ \text{ a.e. } t \\ u(0) = u_0, \quad \sqrt{\varepsilon}u'(0) = u_1 \end{cases}$$

admits at least one solution $u_\varepsilon \in \mathcal{V}$ *such that* $u_\varepsilon' \in \mathcal{W}$.
There exists $u \in L^\infty(0, T; V)$ *with* $u' \in \mathcal{W}$ *and the following convergences hold*

$$\begin{aligned} &u_\varepsilon \longrightarrow u \ \text{ weakly* in } \ L^\infty(0, T; V) \\ &u_\varepsilon' \longrightarrow u' \ \text{ weakly in } \ \mathcal{V} \\ &\sqrt{\varepsilon}u_\varepsilon' \longrightarrow 0 \ \text{ weakly* in } \ L^\infty(0, T; H) \\ &\varepsilon u_\varepsilon'' \longrightarrow 0 \ \text{ weakly in } \ \mathcal{V}^* \end{aligned}$$

as $\varepsilon \longrightarrow 0$*. Moreover, the limit function* u *is a solution to the problem*

$$\begin{cases} A(t)u'(t) + Bu(t) + M^*\partial J(t, Mu(t)) \ni f(t) \ \text{ a.e. } t \in (0, T) \\ u(0) = u_0. \end{cases}$$

Application to quasi-static viscoelastic contact

We consider the mechanical problem of quasi-static viscoelastic contact with nonmonotone normal compliance and friction: find a displacement field $\boldsymbol{u} \colon Q \longrightarrow \mathbb{R}^d$ such that

$$-\mathrm{Div}\,\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \ \text{ in } Q$$

$$\boldsymbol{\sigma}(t) = \mathcal{C}(t)\varepsilon(\boldsymbol{u}'(t)) + \mathcal{G}\varepsilon(\boldsymbol{u}(t)) \ \text{ in } Q$$

$$\boldsymbol{u}(t) = 0 \ \text{ on } \Gamma_D \times (0,T)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_1(t) \ \text{ on } \Gamma_N \times (0,T)$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, u_\nu) \ \text{ on } \Gamma_C \times (0,T)$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(t, \boldsymbol{u}_\tau) \ \text{ on } \Gamma_C \times (0,T)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \ \text{ in } \Omega.$$

Weak solutions to the mechanical problem

We obtain the following hemivariational inequality: find $u\colon (0,T) \to V$ such that $u \in L^\infty(0,T;V)$, $u' \in \mathcal{V}$ and

$$\begin{cases} \langle A(t)u'(t) + Bu(t), v\rangle + \displaystyle\int_{\Gamma_C} \Big(j_\nu^0(t, u_\nu; v_\nu) + j_\tau^0(t, u_\tau; v_\tau) \Big)\, d\Gamma \geq \\ \qquad\qquad \geq \langle f(t), v\rangle \ \text{ a.e. } t \in (0,T), \text{ for all } v \in V \\ u(0) = u_0 \end{cases}$$

where

$$V = \{\, v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \,\}.$$

Under natural hypotheses, applying Theorem 11, we deduce the existence of at least one solution to the hemivariational inequality with the following regularity $u \in L^\infty(0,T;V)$ with $u' \in \mathcal{V}$.

More details can be found in recent papers:

S. Migórski, A. Ochal and M. Sofonea, *History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics*, **Nonlinear Anal. Real World Appl., 12 (2011), 3384–3396**

S. Migórski and A. Ochal, *Quasistatic hemivariational inequality via vanishing acceleration approach*, **SIAM J. Math. Anal., 41 (2009), 1415–1435**

and in a forthcoming book:

S. Migórski, A. Ochal and M. Sofonea, ***Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems***, **Springer Science, Business Media, 2012.**

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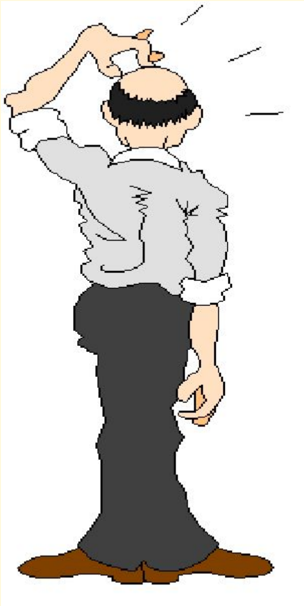
Questions ?



Thank you very much !

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