A finite element solution for the fractional equation

Petra Nováčková, Tomáš Kisela

ÚM FSI VUT v Brně

Nemecká 2011

Contents

Basic Definition

- Left fractional integral of $f\,(x)$

$$ {}_a\mathbf{D}_x^{-\nu} f(x) = \int_a^x \frac{(x-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\ \mathrm{d}\xi,\ \ x \in [a,b) $$

Basic Definition

- Left fractional integral of $f\,(x)$

$$ {}_a\mathbf{D}_x^{-\nu} f(x) = \int_a^x \frac{(x-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\ \mathrm{d}\xi,\ \ x \in [a,b) $$

- Left fractional derivative of $f\,(x)$

$$ {}_a\mathbf{D}_x^{\nu} f(x) = \frac{d^m}{dx^m} \int_a^x \frac{(x-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\ \mathrm{d}\xi,\ \ x \in [a,b) $$

Basic Definition

- Left fractional integral of $f(x)$

$${}_a\mathbf{D}_x^{-\nu} f(x) = \int_a^x \frac{(x-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\ \mathrm{d}\xi,\ \ x \in [a,b)$$

- Left fractional derivative of $f(x)$

$${}_a\mathbf{D}_x^{\nu} f(x) = \frac{d^m}{dx^m} \int_a^x \frac{(x-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\ \mathrm{d}\xi,\ \ x \in [a,b)$$

- Riesz fractional integral of $f(x)$ is defined by the expression

$${}_0\mathbf{D}_1^{-\nu} y(x) = \frac{1}{2} \left({}_0\mathbf{D}_x^{-\nu} y(x) + {}_x\mathbf{D}_1^{-\nu} y(x) \right)\ .$$

Problem

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}\,{}_0\mathbf{D}_{1,x}^{-\nu}u(x,t) + f(x,t)\,, \quad x \in (0,1)\,,\ t \in (0,T)\,,$$

$$\begin{aligned} u(x,0) &= g(x)\,, & x &\in (0,1)\,,\\ \frac{\partial}{\partial x}{}_0\mathbf{D}_{1,x}^{-\nu}u(x,t)\Big|_{x=0} &= 0\,, & t &\in (0,T)\,,\\ \frac{\partial}{\partial x}{}_0\mathbf{D}_{1,x}^{-\nu}u(x,t)\Big|_{x=1} &= 0\,, & t &\in (0,T)\,. \end{aligned}$$

Time Discretization

Time derivative $\frac{\partial}{\partial t}u(x,t)$ is replaced by $\frac{1}{\tau}\left(\tilde{u}^k(x) - \tilde{u}^{k-1}(x)\right)$.

Time Discretization

Time derivative $\frac{\partial}{\partial t} u(x,t)$ is replaced by $\frac{1}{\tau}\left(\tilde{u}^k(x) - \tilde{u}^{k-1}(x)\right)$.

In the weak formulation of time semi-discretized problem we are looking for such functions $\tilde{u}^k$ for which holds:

$$\frac{1}{\tau}\int_0^1 \tilde{u}^k v \,\mathrm{d}x + \int_0^1 \frac{\partial}{\partial x} {}_0\mathbf{D}_1^{-\nu} \tilde{u}^k v' \,\mathrm{d}x = \int_0^1 f^k v \,\mathrm{d}x + \frac{1}{\tau}\int_0^1 \tilde{u}^{k-1} v \,\mathrm{d}x,$$
$$\tilde{u}^0 = g,$$

for all v for which all integrals are properly defined.

Space Discretization

The space partition of the interval (0,1) is equidistant with step $h = 1/N$.

Space Discretization

The space partition of the interval (0,1) is equidistant with step $h = 1/N$.

Let us consider the approximative solution in the form

$$U^k(x) = \sum_{j=0}^{N} U_j^k w_j(x),$$

Space Discretization

The space partition of the interval (0,1) is equidistant with step $h = 1/N$.

Let us consider the approximative solution in the form

$$U^k(x) = \sum_{j=0}^{N} U_j^k w_j(x),$$

The problems with choice of basis and test functions are:

Space Discretization

The space partition of the interval (0,1) is equidistant with step $h = 1/N$.

Let us consider the approximative solution in the form

$$U^k(x) = \sum_{j=0}^{N} U_j^k w_j(x),$$

The problems with choice of basis and test functions are:

- Stable solutions grows near the boundary.

Space Discretization

The space partition of the interval (0,1) is equidistant with step $h = 1/N$.

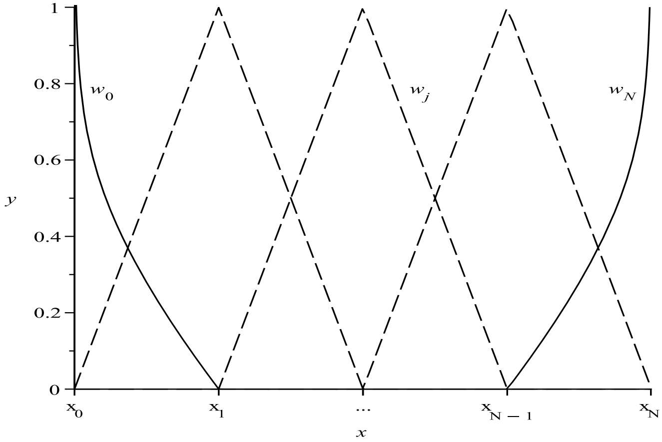
Let us consider the approximative solution in the form

$$U^k(x) = \sum_{j=0}^{N} U_j^k w_j(x),$$

The problems with choice of basis and test functions are:

- Stable solutions grows near the boundary.
- We want the integrals which appear during the derivation of FEM scheme to be analytically computable.

Basis Functions



Basis functions for $\nu = 0.8$.

Matrix of the System

- Mass matrix of the system is not spare three-diagonal matrix, but it is a full matrix

Matrix of the System

- Mass matrix of the system is not spare three-diagonal matrix, but it is a full matrix
- Elements on the diagonal (except the first and the last one) are positive and in absolute value are larger than the others in its row.

Matrix of the System

- Mass matrix of the system is not spare three-diagonal matrix, but it is a full matrix
- Elements on the diagonal (except the first and the last one) are positive and in absolute value are larger than the others in its row.
- Elements outside the diagonal are negative and rapidly decreasing to zero.

Matrix of the System

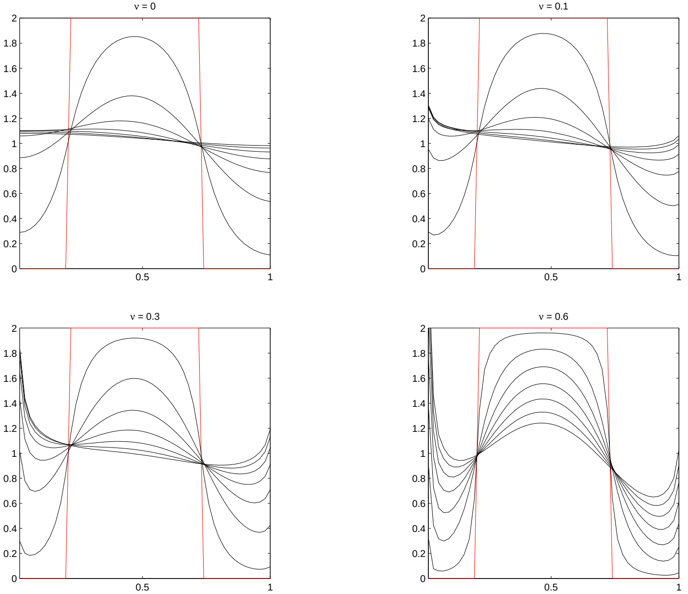
- Mass matrix of the system is not spare three-diagonal matrix, but it is a full matrix
- Elements on the diagonal (except the first and the last one) are positive and in absolute value are larger than the others in its row.
- Elements outside the diagonal are negative and rapidly decreasing to zero.
- For $\nu = 0$ mass the matrix becomes three-diagonal.

Example

Values of parameters are:

- order of the derivation ν is successively $0, 0.1, 0.3, 0.6;$
- final time $T = 0.2$
- time step $\tau = 0.01$
- space steps $h = 0.02$
- problem is without source term: $f(x,t) \equiv 0$
- the initial condition is

$$g(x) = \begin{cases} 2 & \text{for} \quad x \in (0.2; 0.7), \\ 0 & \text{pro} \quad x \notin (0.2; 0.7). \end{cases}$$


ν = 0
ν = 0.1
ν = 0.3
ν = 0.6
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.5
1

References

Tomáš Kisela.
Applications of the fractional calculus: On a discretization of fractional diffusion equation in one dimension.
Communications, 12(1):5–11, 2010.

Kenneth S. Miller and Bertram Ross.
An Introduction to the Fractional Calculus and Fractional Differential Equations.
John Wiley & Sons, 1st edition, 1993.

K.B. Oldham and J. Spanier.
The fractional calculus: theory and applications of differentiation and integration to arbitrary order.
Dover books on mathematics. Dover Publications, 2006.

Karel Rektorys.
Metoda časové diskretizace a parciální diferenciální rovnice: účinná a široce aplikovatelná metoda řešení parciálních diferenciálních rovnic obsahujících čas.
Teoretická knižnice inženýra. SNTL, 1985.

John Paul Roop and Vincent J. Ervin.
Variational formulation for the fractional advection dispersion equation.
Numerical Methods for Partial Differential Equations, 48:558–576, 2006.

Thank you!

Research is sponsored by Brno University of Technology, specific research, project no. FSI-S-11-3;
Author's addresses: *Petra Nováčková*, Institute of Mathematics, Brno University of Technology, Technická 2, CZ-616 69, Brno, Czech Republic, e-mail: petra.novackova@gmail.com. The author accepts scholarship "Brno Ph.D." talent provided by the statutory city of Brno;
Tomáš Kisela, Institute of Mathematics, Brno University of Technology, Technická 2, CZ-616 69 Brno, Czech Republic, e-mail: kisela.tomas@gmail.com.