Some Remarks on Operator Generalized Effect Algebras

Marcel Polakovič

Some Remarks on Operator Generalized Effect Algebras

Marcel Polakovič

The talk is based on paper: M.Polakovič, Z Riečanová, "Generalized Effect Algebras of Positive Operators Densely Defined on Hilbert Spaces", Int. J. Theor. Phys. (2011) 50: 1167-1174

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?
- Quantum structures: quantum logics, effect algebras etc.

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?
- Quantum structures: quantum logics, effect algebras etc.
- Quantum mechanics - observables are (unbounded) operators on Hilbert space

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?
- Quantum structures: quantum logics, effect algebras etc.
- Quantum mechanics - observables are (unbounded) operators on Hilbert space
- The first example of effect algebra (different from Boolean algebra) - Foulis and Bennett (1994) - the set of Hilbert space effects $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?
- Quantum structures: quantum logics, effect algebras etc.
- Quantum mechanics - observables are (unbounded) operators on Hilbert space
- The first example of effect algebra (different from Boolean algebra) - Foulis and Bennett (1994) - the set of Hilbert space effects $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$
- partial operation $\oplus$ on $\mathcal{E}(\mathcal{H})$: $A \oplus B$ is defined and equal to $A + B$ iff $A + B \leq I$.

- Question (M.Znojil): What is the relationship between the Hilbert space quantum mechanics and quantum structures?
- Quantum structures: quantum logics, effect algebras etc.
- Quantum mechanics - observables are (unbounded) operators on Hilbert space
- The first example of effect algebra (different from Boolean algebra) - Foulis and Bennett (1994) - the set of Hilbert space effects $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$
- partial operation $\oplus$ on $\mathcal{E}(\mathcal{H})$: $A \oplus B$ is defined and equal to $A + B$ iff $A + B \leq I$.
- $\mathcal{E}(\mathcal{H})$ satisfies the conditions of the following definition:

Definition (Foulis and Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0,1 are two distinct elements and $\oplus$ is a partial operation on E for which

- (E1): $x \oplus y = y \oplus$ x if $x \oplus y$ is defined
- (E2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (E3): for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$
- (E4): if $1 \oplus x$ is defined then $x = 0$

Definition (Foulis and Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0,1 are two distinct elements and $\oplus$ is a partial operation on E for which

- (E1): $x \oplus y = y \oplus x$ if $x \oplus y$ is defined
- (E2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (E3): for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$
- (E4): if $1 \oplus x$ is defined then $x = 0$

- So the first example of effect algebra was modelled by operators in Hilbert space

Definition (Foulis and Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0,1 are two distinct elements and $\oplus$ is a partial operation on E for which

- (E1): $x \oplus y = y \oplus x$ if $x \oplus y$ is defined
- (E2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (E3): for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$
- (E4): if $1 \oplus x$ is defined then $x = 0$

- So the first example of effect algebra was modelled by operators in Hilbert space
- Generalizations of effect algebras (without a top element 1) have been studied - generalized effect algebras

Definition

(1) *Generalized effect algebra* $(E; \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions

- (GE1): $x \oplus y = y \oplus$ x if one side is defined
- (GE2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (GE3): if $x \oplus y = x \oplus z$ then $y = z$ (cancellation law)
- (GE4): if $x \oplus y = 0$ then $x = y = 0$
- (GE5): $x \oplus 0 = x$ for all $x \in E$

Definition

(1) *Generalized effect algebra* $(E; \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions

- (GE1): $x \oplus y = y \oplus$ x if one side is defined
- (GE2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (GE3): if $x \oplus y = x \oplus z$ then $y = z$ (cancellation law)
- (GE4): if $x \oplus y = 0$ then $x = y = 0$
- (GE5): $x \oplus 0 = x$ for all $x \in E$

(2) Define a binary relation $\leq$ on E by $x \leq y$ iff for some $z \in E$, $x \oplus z = y$

Definition

(1) *Generalized effect algebra* $(E; \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions

- (GE1): $x \oplus y = y \oplus$ x if one side is defined
- (GE2): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
- (GE3): if $x \oplus y = x \oplus z$ then $y = z$ (cancellation law)
- (GE4): if $x \oplus y = 0$ then $x = y = 0$
- (GE5): $x \oplus 0 = x$ for all $x \in E$

(2) Define a binary relation $\leq$ on E by $x \leq y$ iff for some $z \in E$, $x \oplus z = y$

(3) $Q \subseteq E$ is a *sub-generalized effect algebra* iff out of elements $x, y, z \in E$ with $x \oplus y = z$ at least two are in Q then $x, y, z \in Q$

- Every sub-generalized effect algebra of E is a generalized effect algebra in its own right
- Every effect algebra is a generalized effect algebra in the natural sense

- Every sub-generalized effect algebra of E is a generalized effect algebra in its own right
- Every effect algebra is a generalized effect algebra in the natural sense
- (Foulis and Bennett, 1994) - the effect algebra $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$ was modelled by (bounded) operators on Hilbert space. (It is important for mathematical description of unsharp measurement in quantum mechanics.)
- Later, effect algebras and generalized effect algebras were modelled by objects of different kind (e.g. fuzzy sets) or regarded as abstract structures

- Every sub-generalized effect algebra of E is a generalized effect algebra in its own right
- Every effect algebra is a generalized effect algebra in the natural sense
- (Foulis and Bennett, 1994) - the effect algebra $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$ was modelled by (bounded) operators on Hilbert space. (It is important for mathematical description of unsharp measurement in quantum mechanics.)
- Later, effect algebras and generalized effect algebras were modelled by objects of different kind (e.g. fuzzy sets) or regarded as abstract structures
- The aim of the present work is to show another examples of generalized effect algebras modelled by (possibly unbounded) operators on Hilbert space. (So the introductory question about the relationship between quantum mechanics and quantum structures is only an inspiration.)

- $\mathcal{H}$ - complex Hilbert space
- A linear operator A on $\mathcal{H}$ with domain $D(A)$ is densely defined if $\overline{D(A)} = \mathcal{H}$. A is positive if $(Ax, x) \geq 0$ for all $x \in D(A)$.

- $\mathcal{H}$ - complex Hilbert space
- A linear operator A on $\mathcal{H}$ with domain $D(A)$ is densely defined if $\overline{D(A)} = \mathcal{H}$. A is positive if $(Ax, x) \geq 0$ for all $x \in D(A)$.

Theorem

Let $\mathcal{H}$ be a complex Hilbert space and let $D \subseteq \mathcal{H}$ be a linear subspace dense in $\mathcal{H}$. Let

$$\mathcal{G}_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \text{ is a positive linear operator defined on } D\}$$

Then $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra where 0 is the null operator and $\oplus$ is the usual sum of operators defined on D. In this case $\oplus$ is a total operation.

- $\mathcal{H}$ - complex Hilbert space
- A linear operator A on $\mathcal{H}$ with domain $D(A)$ is densely defined if $\overline{D(A)} = \mathcal{H}$. A is positive if $(Ax, x) \geq 0$ for all $x \in D(A)$.

Theorem

Let $\mathcal{H}$ be a complex Hilbert space and let $D \subseteq \mathcal{H}$ be a linear subspace dense in $\mathcal{H}$. Let

$$\mathcal{G}_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \textit{ is a positive linear operator defined on } D\}$$

Then $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra where 0 is the null operator and $\oplus$ is the usual sum of operators defined on D. In this case $\oplus$ is a total operation.

- So all positive operators defined on a fixed dense subspace in $\mathcal{H}$ form a generalized effect algebra with the operation of the usual operator sum.

- The most difficult condition to prove was $(GE4)$: if $A \oplus B = 0$ then $A = B = 0$.

- The most difficult condition to prove was $(GE4)$: if $A \oplus B = 0$ then $A = B = 0$.
- The operators $A \in \mathcal{G}_D(\mathcal{H})$ may be unbounded. So $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is an example of generalized effect algebra modelled by (possibly unbounded) operators on Hilbert space $\mathcal{H}$.

- The most difficult condition to prove was $(GE4)$: if $A \oplus B = 0$ then $A = B = 0$.
- The operators $A \in \mathcal{G}_D(\mathcal{H})$ may be unbounded. So $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is an example of generalized effect algebra modelled by (possibly unbounded) operators on Hilbert space $\mathcal{H}$.
- The next Theorem deals with bounded operators

Theorem

Let $\mathcal{H}$ be a complex Hilbert space and $D \subseteq \mathcal{H}$ be a dense linear subspace of $\mathcal{H}$. Then the set of all bounded positive linear operators on D form a sub-generalized effect algebra of $\mathcal{G}_D(\mathcal{H})$ with respect to usual addition of operators, which in this case is a total operation.

- The most difficult condition to prove was $(GE4)$: if $A \oplus B = 0$ then $A = B = 0$.
- The operators $A \in \mathcal{G}_D(\mathcal{H})$ may be unbounded. So $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is an example of generalized effect algebra modelled by (possibly unbounded) operators on Hilbert space $\mathcal{H}$.
- The next Theorem deals with bounded operators

Theorem

Let $\mathcal{H}$ be a complex Hilbert space and $D \subseteq \mathcal{H}$ be a dense linear subspace of $\mathcal{H}$. Then the set of all bounded positive linear operators on D form a sub-generalized effect algebra of $\mathcal{G}_D(\mathcal{H})$ with respect to usual addition of operators, which in this case is a total operation.

- So bounded operators (with the usual operator addition) form a generalized effect algebra. (It is also possible to choose $D = \mathcal{H}$.)

- Now we show a generalized effect algebra including all (also unbounded) positive linear operators densely defined on $\mathcal{H}$, without fixed domain $D \subseteq \mathcal{H}$.

- Now we show a generalized effect algebra including all (also unbounded) positive linear operators densely defined on $\mathcal{H}$, without fixed domain $D \subseteq \mathcal{H}$.
- If A, B are linear operators with domains $D(A), D(B)$, $D(A) \subseteq D(B)$ then $B|_{D(A)}$ is the restriction of B to $D(A)$. $A + B$ means the usual addition of operators.

- Let $\mathcal{V}(\mathcal{H})$ denotes the set of all positive linear operators on $\mathcal{H}$ with the domain $D(A) = \mathcal{H}$ if A is bounded and with $\overline{D(A)} = \mathcal{H}$ if A is unbounded.

- Let $\mathcal{V}(\mathcal{H})$ denotes the set of all positive linear operators on $\mathcal{H}$ with the domain $D(A) = \mathcal{H}$ if A is bounded and with $\overline{D(A)} = \mathcal{H}$ if A is unbounded.

Theorem

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. Let $\oplus$ be a partial binary operation on $\mathcal{V}(\mathcal{H})$ defined by $A \oplus B = A + B$ with $D(A \oplus B) = \mathcal{H}$ for any bounded $A, B \in \mathcal{V}(\mathcal{H})$ and $A \oplus B = B \oplus A = A + B|_{D(A)}$ with $D(A \oplus B) = D(A)$ if A is unbounded and B is bounded. Then $(\mathcal{V}(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra.

Moreover, the set $\mathcal{B}_p(\mathcal{H})$ of all bounded positive linear operators defined on $\mathcal{H}$ is a sub-generalized effect algebra of $\mathcal{V}(\mathcal{H})$ with respect to inherited $\oplus$-operation, which becomes total on $\mathcal{B}_p(\mathcal{H})$.