ON THE REFLEXIVITY AND TRANSITIVITY OF THE TOEPLITZ OPERATORS ON THE UPPER HALF-PLANE

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Let H be a Hilbert space and $\mathcal{B}(H)$ the algebra of all linear and bounded operators on H. A *reflexive closure* of a subspace $\mathcal{S} \subset \mathcal{B}(H)$ is given by

$$\operatorname{ref}\mathcal{S} = \{B \in \mathcal{B}(H) : Bh \in \overline{\mathcal{S}h} \ \text{ for all } h \in H\}.$$

It is clear that $\mathcal{S} \subset \operatorname{ref}\mathcal{S} \subset \mathcal{B}(H)$. Subspace $\mathcal{S} \subset \mathcal{B}(H)$ is called *reflexive* if $\operatorname{ref}\mathcal{S}$ is as small as it can be i.e. $\operatorname{ref}\mathcal{S} = \mathcal{S}$ and it is called *transitive* if $\operatorname{ref}\mathcal{S}$ is as big as it can be i.e. $\operatorname{ref}\mathcal{S} = \mathcal{B}(H)$.

A dichotomy (reflexive or transitive) behavior of subspaces of Toeplitz operators on the Hardy space on the unit disk was shown in [1]. There were presented conditions where the subspace is reflexive (not transitive) by existence of a Toeplitz operator in the predual with logarithmicaly summable symbol. Our aim is to find parallel characterization of subspaces of Toeplitz operators on the Hardy space on the upper half-plane.

The Hardy space $H^2(\mathbb{C}_+)$ on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the space of all analytic functions $F \colon \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|F\|_{H^2(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |F(x+iy)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Since, for any function from $H^2(\mathbb{C}_+)$, there are limits a.e. from the upper half-plane to the real line, $H^2(\mathbb{C}_+)$ can be regarded as a subspace of $L^2(\mathbb{R})$ (see [2]). Let $P_{H^2(\mathbb{C}_+)}$ denote the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$.

For each $\Phi \in L^\infty(\mathbb{R})$ a *Toeplitz operator* with symbol Φ is an operator $T_\Phi \in \mathcal{B}(H^2(\mathbb{C}_+))$ defined by

$$T_\Phi F = P_{H^2(\mathbb{C}_+)}(\Phi F), \ F \in H^2(\mathbb{C}_+).$$

Let $\mathcal{T}(\mathbb{C}_+)$ denote the space of all Toeplitz operators.

The main result is

Theorem. *Suppose $\mathcal{F} \subset \mathcal{T}(\mathbb{C}_+)$ is a weak*–closed subspace. Then the following statements are equivalent.*

(1) *$\mathcal{F}$ is not transitive.*
(2) *There is a function $F \colon \mathbb{R} \to \mathbb{C}$ such that $F \in L^1(\mathbb{R})$, $\log|F| \in L^1\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$ and $\int_{\mathbb{R}} \Phi F dt = 0$ for all $T_\Phi \in \mathcal{F}$.*
(3) *$\mathcal{F}$ is reflexive.*

Since the weak* topology plays important role in the proof the classical isomorphism between L^1 spaces on the real line and on the unit circle have to be redefined to get the proper relationship between weak* topology on the Toeplitz operators on the Hardy space on the upper half-plane and Toeplitz operators on the Hardy space on the unit disk.

References

[1] E. A. Azoff, M. Ptak, *A Dichotomy for Linear Spaces of Toeplitz Operators*, J. Funct. Anal. 156, 411-428 (1998).
[2] P. Koosis, *Introduction to H_p Spaces*, Cambridge University Press, Cambridge 1980.

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