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# REFLEXIVITY DEFECT OF THE KERNEL OF SOME ELEMENTARY OPERATORS

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## 1. Introduction

Let $\mathcal{X}$ be a finite dimensional complex Banach space and $\mathcal{S}$ be a nonempty subset of $L(\mathcal{X})$, the space of all linear operators on $\mathcal{X}$. Let $k$ be a positive integer. Define the $k$-reflexive cover of $\mathcal{S}$ to be the space

(1)
$$\mathrm{Ref_k}\mathcal{S} = \{T \in L(\mathcal{X}) : \forall \varepsilon > 0,\ \forall x_1, \ldots, x_k \in \mathcal{X} : \exists S \in \mathcal{S} :\ \|Tx_i - Sx_i\| < \varepsilon,\ \ i = 1, \ldots, k\}.$$

One can see that $\mathrm{Ref_k}\mathcal{S}$ is a linear subspace of $L(\mathcal{X})$. A linear subspace $\mathcal{S}$ is said to be $k$-reflexive if $\mathrm{Ref_k}\mathcal{S} = \mathcal{S}$. The $k$-reflexivity defect is defined by $\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref_k}\mathcal{S}) - \dim(\mathcal{S})$. It is easy to see that the following lemma holds.

**Lemma 1.1.** *Suppose that $\mathcal{X} = \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_N$ is a decomposition of $\mathcal{X}$. Let $\mathcal{S} \subseteq L(\mathcal{X})$ be a linear subspace and for each $S \in \mathcal{S}$ let $[S_{ij}]$ be the block matrix representing $S$ with respect to the above decomposition of $\mathcal{X}$. For each pair of indices $i$, $j$ denote by $\mathcal{S}_{ij}$ the subspace of $L(\mathcal{X}_j, \mathcal{X}_i)$ consisting of $S_{ij}$ defined above. Let $k \in \mathbb{N}$ be a positive integer. Then each $T \in \mathrm{Ref_k}\mathcal{S}$ has the block matrix representation $[T_{ij}]$ where $T_{ij} \in \mathrm{Ref_k}\mathcal{S}_{ij}$. For the $k$-reflexivity defect one has $\mathrm{rd}_k(\mathcal{S}) = \sum_{i,j=1}^{N} \mathrm{rd}_k(\mathcal{S}_{ij})$; in particular, $\mathcal{S}$ is reflexive if and only if $\mathcal{S}_{ij}$ is reflexive for every $i$, $j \in \{1, \ldots, N\}$.*

Let $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ be arbitrary $k$-tuples of operators on $\mathcal{X}$. The elementary operator on $L(\mathcal{X})$ with coefficients $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ is defined by

$$\Delta(T) = B_1TA_1 + B_2TA_2 + \ldots + B_kTA_k, \qquad \big(T \in L(\mathcal{X})\big). \tag{2}$$

It is easy to see that the kernel of $\Delta$ is a $k$-reflexive subspace of $L(\mathcal{X})$, i.e., $\mathrm{rd}_k(\ker\Delta) = 0$. Hence, it is reasonable to ask whether $\mathrm{rd}_j(\ker\Delta)$ can be determined for $j < k$. In what follows, we are interested in the 1-reflexivity defect of the kernel of elementary operators of the form $\Delta(T) = B_1TA_1 - B_2TA_2$, $T \in L(\mathcal{X})$, where $A_1$, $A_2$ and $B_1$, $B_2$ are linearly independent. To shorten the notation we will write $\mathrm{rd}(\ker\Delta)$ instead of $\mathrm{rd}_1(\ker\Delta)$. In Section 2 we consider some special types of such an elementary operator. In Section 3 we are dealing with the images of some special types of elementary operators.

## 2. Elementary operators of length 2

Because the $k$-reflexivity defect is preserved by similarity transformations, one can assume that $\mathcal{X} = \mathbb{C}^n$ where $n \in \mathbb{N}$. Thus, $L(\mathcal{X})$ may be identified with $\mathbb{M}_n$, the algebra of all $n$-by-$n$ complex matrices. Let $A$, $B \in \mathbb{M}_n$ be arbitrary matrices. Define the generalized derivation on $\mathbb{M}_n$ with coefficients $A$ and $B$ by $\delta(T) = BT - TA$, $T \in \mathbb{M}_n$. By a result of Zajac in [2], $\ker\delta$ is reflexive if and only if all roots of the greatest common divisor of the minimal polynomials $m_A$ and $m_B$ of $A$ and $B$, respectively, are simple.

For $k \in \mathbb{N}$, let $J_k$ denote the Jordan block of order $k$, i.e.,

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ be the Jordan canonical form of $A$, where $\sum_{i=1}^{N} p_i = n$ and $\lambda_1, \ldots, \lambda_N$ are not necessarily distinct eigenvalues of $A$. Similarly, let $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ be the Jordan canonical form of $B$, where $\sum_{i=1}^{M} r_i = n$ and $\mu_1, \ldots, \mu_M$ are not necessarily distinct eigenvalues of $B$. Let $R(i, j)$ be a nonnegative integer defined by

$$R(i, j) := \begin{cases} \frac{1}{2} \min\{r_i, p_j\} (\min\{r_i, p_j\} - 1) & \text{if } \mu_i = \lambda_j \\ 0 & \text{otherwise.} \end{cases}$$

The following, a little more general result than the one in [2], which has been mentioned above, can be obtained, cf. [1] and [3].

**Proposition 2.1.** *With the above notation, the reflexivity defect of* $\ker \delta$ *can be expressed as*

$$\mathrm{rd}(\ker \delta) = \sum_{i=1}^{M} \sum_{j=1}^{N} R(i, j).$$

*In particular,* $\ker \delta$ *is a reflexive space if and only if* $\deg(\gcd(m_A(z), m_B(z))) \leq 1$.

Let $A, B \in \mathbb{M}_n$ be as before and let $\epsilon$ be the elementary operator on $\mathbb{M}_n$ defined by $\epsilon(T) = BTA - T$, $T \in \mathbb{M}_n$. Then the following holds.

**Proposition 2.2.** *If* $1 \notin \sigma(A)\,\sigma(B)$, *then* $\ker \epsilon$ *is reflexive. Otherwise*

$$\mathrm{rd}(\ker \epsilon) = \sum_{\mu_i \lambda_j = 1} \frac{1}{2} \min\{r_i, p_j\} (\min\{r_i, p_j\} - 1).$$

*Proof.* Since $A$ is similar to $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ and $B$ is similar to $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ Lemma 1.1 yields that $\mathrm{rd}(\ker \epsilon) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{rd}(\ker \epsilon_{ij})$, where $\epsilon_{ij}(T) = (\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) - T$ is the elementary operator acting on the space of $r_i$-by-$p_j$ matrices. If $T$ satisfies the equation $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T$, then $T$ is an eigenvector of the Kronecker product $(\lambda_j + J_{p_j})^{\mathrm{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1. Since $\sigma\left((\lambda_j + J_{p_j})^{\mathrm{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$ we can conclude that if $\lambda_j \mu_i \neq 1$, then $\ker \epsilon_{ij} = \{0\}$. Otherwise, if $\lambda_j \mu_i = 1$, then $\mu_i + J_{r_i}$ and $\lambda_j + J_{p_j}$ are invertible and hence $\ker \epsilon_{ij} = \ker \tilde{\epsilon}_{ij}$, where $\tilde{\epsilon}_{ij}$ is a generalized derivation of the form $\tilde{\epsilon}_{ij}(T) = (\mu_i + J_{r_i})T - T(\lambda_j + J_{p_j})^{-1}$. Since inverting matrices preserves sizes of Jordan blocks the result follows by Proposition 2.1. ☐

## 3. On $k$-reflexivity defect of the image of some elementary operators

Let $A, B \in \mathbb{M}_n$ be arbitrary matrices and let $\tau$ be an elementary operator defined by $\tau(T) = BTA$ for $T \in \mathbb{M}_n$. It is easy to see that the kernel and the image of $\tau$ are reflexive spaces. Considering this and the fact that the kernel of an elementary operator of the form (2) is $k$-reflexive one asks whether the same holds for image of such an operator. We will show that this is not the case even if $\Delta$ is a generalized derivation. First we introduce some notation. The annihilator of a nonempty subset $\mathcal{S} \subseteq \mathbb{M}_n$ is defined by $\mathcal{S}_\perp = \{C \in \mathbb{M}_n : \mathrm{tr}(CS) = 0 \text{ for all } S \in \mathcal{S}\}$, where $\mathrm{tr}(\cdot)$ is the trace functional.

**Lemma 3.1.** *Let* $\Delta$ *be an elementary operator on* $\mathbb{M}_n$ *with coefficients* $(A_1, \ldots, A_k)$ *and* $(B_1, \ldots, B_k)$, *defined by* $\Delta(T) = B_1TA_1 + B_2TA_2 + \ldots + B_kTA_k$. *Then there exists an elementary operator* $\tilde{\Delta}$ *such that* $(\mathrm{im}\,\Delta)_\perp = \ker \tilde{\Delta}$.

*Proof.* Define $\tilde{\Delta}(T) = A_1TB_1 + \ldots + A_kTB_k$ for $T \in \mathbb{M}_n$. If $T$ is an arbitrary matrix, then $\operatorname{tr}(\Delta(T)C) = \operatorname{tr}(T(A_1CB_1 + \ldots + A_kCB_k))$ and therefore $C \in (\operatorname{im}\Delta)_\perp$ if and only if $\tilde{\Delta}(C) \in (\mathbb{M}_n)_\perp = \{0\}$, that is, $C \in \ker\tilde{\Delta}$. □

Denote by $F_k$ the set of elements in $\mathbb{M}_n$ of rank $k$ or less. It is well known that $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$. In the following example we show that there exists a generalized derivation $\delta$ on $\mathbb{M}_3$ such that $\operatorname{im}\delta$ is not 2-reflexive.

**Example 3.2.** Define $\delta(T) = J_3T - TJ_3$ for $T \in \mathbb{M}_3$. Obviously, $\operatorname{im}\delta$ is 3-reflexive. By Lemma 3.1, we get

$$(\operatorname{im}\delta)_\perp = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : \, a, \, b, \, c \in \mathbb{C} \right\}.$$

Since $(\operatorname{im}\delta)_\perp \cap F_2 \subsetneq (\operatorname{im}\delta)_\perp$ we have $\operatorname{im}\delta \subsetneq \operatorname{Ref}_2(\operatorname{im}\delta)$ and thus, by (1), $\operatorname{im}\delta$ is not 2-reflexive. Moreover, we have $\operatorname{rd}(\operatorname{im}\delta) = 2$ and $\operatorname{rd}_2(\operatorname{im}\delta) = 1$.

## References


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