On a dynamic contact problem for a geometrically nonlinear viscoelastic shell

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Outline

1. **Hyperbolic (dynamic) contact problems**
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2. **Notations and geometrical assumptions**
3. **Contact of a viscoelastic shell**
   - Contact of the shell free on the boundary
   - Solving by penalization

Vibrating string against an obstacle

Much less papers comparing with static and quasistatic problems

L. Amerio and G. Prouse, Study of the motion of a string vibrating against an obstacle, *Rend. Mat.* 2 (1975), 563-585.

C. Citrini, The energy theorem in the impact of a string vibratng against an obstacle, *Atti Accad. Naz. Linzei Rend.* 62 (1977), 143-149.

M. Schatzman, A hyperbolic problem of second order with unilateral constraints: The vibrating string with a concave obstacle, *Journal of Math. Anal. and Appl.* 73, (1980), 138-191.

Contact problems for membranes and plates

Hilbert space approach:
K. Maruo, Existence of solutions of some nonlinear wave equations, *Osaka Math. J.* 22, (1984), 21-30.

Boundary contact:
J.U. Kim, A boundary thin obstacle problem for a wave equation, *Comm.Part.Diff.Eqs.* 14, (1989), 1011-1026.

Viscoelastic membrane:

J. Jarušek, J. Málek, J. Nečas, and V. Švěrák, Variational inequality for a viscous drum vibrating in the presence of an obstacle, *Rend. Mat.* 12, (1992), 943-958.

Dynamic contact of geometrically nonlinear plates

Viscoelastic short memory plate:
I.Bock and J. Jarušek: Unilateral dynamic contact of viscoelastic von Kármán plates. *Advances in Math. Sci. and Appl.* **16** (2006), 175–187.
Singular memory plate:
I.B. and J.J.: Unilateral dynamic contact of von Kármán plates with singular memory. *Applications of Math.* **52** (2007), 515–527.
Elastic plate:
I.B. and J.J.: Solvability of dynamic contact problems for elastic von Kármán plates. *SIAM J. Math. Anal.* **41** (2009), 37–45.

Notations

$\Omega \subset \mathbb{R}^2$ is a bounded domain with a boundary Γ.
$I \equiv (0, T)$ a time interval, $Q = I \times \Omega$, $S = I \times \Gamma$.

The unit outer normal vector $\boldsymbol{n} = (n_1, n_2)$,
$\boldsymbol{\tau} = (-n_2, n_1)$ the unit tangent vector.

Partial derivatives

$$\frac{\partial}{\partial s} \equiv \partial_s, \ \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr}, \ \partial_i = \partial_{x_i}, \ i = 1, 2, 3,$$
$$\dot{v} = \frac{\partial v}{\partial t}, \ \ddot{v} = \frac{\partial^2 v}{\partial t^2}.$$

Function spaces

$W_p^k(M) \subset L_p(M)$, $k \geq 0$, $p \in [1, \infty]$ the Sobolev spaces.

$\mathring{W}_p^k(M) \subset W_p^k(M)$
the spaces of functions with zero traces on the boundary .

$H^k(M) = W_2^k(M),\ \mathring{H}^k(M) = \mathring{W}_2^k(M),\ H^{-k}(M) = (\mathring{H}^k(M))^*.$

The anisotropic spaces
$W_p^k(M),\ k = (k_1, k_2) \in \mathbb{R}_+^2,\ \mathbb{R}_+ = (0, \infty),$
k_1 is related with the time while k_2 with the space variables.

$C(M),\ C(I; M)$ the spaces of continuous functions.

$\mathcal{H} = L_\infty(I; H^2(\Omega)),\ \mathring{\mathcal{H}} = L_\infty(I; \mathring{H}^2(\Omega)).$

Strain-displacements relations

A shallow isotropic shell occupying the domain

$$\mathcal{A} = \{(x, z) \in R^3 : x = (x_1, x_2) \in \Omega, |z - \mathcal{S}(x)| < h\}$$

with $z = \mathcal{S}(x),\ x \in \Omega$ - a middle surface of a shell.

The displacement $\boldsymbol{u} \equiv (u_i)$.

Moderately large deflections for von Kármán-Donnell model :
Strain tensor

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - x_3 \partial_{ij} u_3;$$
$$k_{12} = 0,\ \text{curvatures } k_{ii} > 0,\ i, j = 1, 2.$$

The important bracket

$$[u, v] \equiv \partial_{11} u \partial_{22} v + \partial_{22} u \partial_{11} v - 2\partial_{12} u \partial_{12} v.$$

Stress-strain relation

$$\sigma_{ij}(\boldsymbol{u}) = \frac{E_1}{1-\mu^2}\partial_t\left((1-\mu)\varepsilon_{ij}(\boldsymbol{u}) + \mu\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})\right) + \frac{E_0}{1-\mu^2}\left((1-\mu)\varepsilon_{ij}(\boldsymbol{u}) + \mu\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})\right).$$

$E_0 > 0$ - the Young modulus of elasticity and
$E_1 > 0$ - the modulus of viscosity,
$a = \frac{h^2}{12}$ - the rotation inertia term, $b = \frac{h^2}{12\varrho(1-\mu^2)}$
$\mu \in \langle 0, \frac{1}{2})$ Poisson ratio.
$2h > 0$ - the shell thickness, $\varrho > 0$ - the density of the material.

The shell free on the boundary is acting under
the perpendicular load $f : Q \mapsto R$
and an unknown contact force g.

Classical formulation

Initial-boundary value problem

for the bending function u and the Airy stress function v

$$\left.\begin{array}{l} \ddot{u} + a\Delta\ddot{u} + b(E_1\Delta^2\dot{u} + E_0\Delta^2 u) - [u,v] - \Delta_k v \\ = f + g, \\ u \geq 0,\ g \geq 0,\ ug = 0, \\ \Delta^2 v = \\ -\left(E_1\partial_t(\frac{1}{2}[u,u] + \Delta_k u) + E_0(\frac{1}{2}[u,u] + \Delta_k u)\right) \end{array}\right\} \text{on } Q, \quad (1)$$

$$\left.\begin{array}{l} u \geq 0,\ \Sigma(u) \geq 0,\ u\Sigma(u) = 0, \\ \mathcal{M}(u) = 0,\ v = \partial_{\boldsymbol{n}} v = 0 \end{array}\right\} \text{on } S, \quad (2)$$

$$u(0,\cdot) = u_0 \geq 0,\ \dot{u}(0,\cdot) = u_1 \text{ on } \Omega, \quad (3)$$

Bilinear form

with the generalized Laplacian

$$\Delta_k u = k_{22}\partial_{11} u + k_{11}\partial_{22} u,\ \ u \in H^2(\Omega).$$

We introduce the bilinear form

$$A(u, y) := \partial_{11} u \partial_{11} y + \partial_{22} u \partial_{22} y$$
$$+\mu(\partial_{11} u \partial_{22} y + \partial_{22} u \partial_{11} y) + 2(1-\mu)\partial_{12} u \partial_{12} y$$

and a cone $\mathcal{C}$ of nonnegative functions

$$\mathcal{C} := \{y \in H^{1,2}(Q);\ \dot{y} \in L_2(I; H^2(\Omega)),\ y \geq 0\}.$$

Variational formulation-inequality for u

Find $\{u, v\} \in \mathcal{C} \times L_2(I; \mathring{H}^2(\Omega))$ such that u fulfils

$$\begin{aligned}
&\int_Q (E_1 A(\dot{u}, y_1 - u) + E_0 A(u, y_1 - u) - a\nabla\dot{u}\cdot\nabla(\dot{y}_1 - \dot{u}) \\
&- \dot{u}(\dot{y}_1 - \dot{u}) - ([u, v] + \Delta_k v)(y_1 - u))\, dx\, dt \\
&+ \int_\Omega (a\nabla\dot{u}\cdot\nabla(y_1 - u) + \dot{u}(y_1 - u))\,(T, \cdot)\, dx \\
&\geq \int_\Omega (a\nabla u_1\cdot\nabla(y_1(0, \cdot) - u_0) + u_1(y_1(0, \cdot) - u_0))\; dx \\
&+ \int_Q f(y_1 - u)\, dx\, dt \;\forall y_1 \in \mathcal{C},
\end{aligned} \tag{4}$$

Variational formulation-equation for v

and v fulfils

$$\int_\Omega \Delta v \Delta y_2 \, dx = - \int_\Omega \Big(E_1 \partial_t (\frac{1}{2}[u,u] + \Delta_k u) + E_0 (\frac{1}{2}[u,u] + \Delta_k u) \Big) y_2 \, dx \quad \forall \, y_2 \in \mathring{H}^2(\Omega). \tag{5}$$

The linear compact operator $L : H^2(\Omega) \to \mathring{H}^2(\Omega)$ defined by

$$\int_\Omega \Delta L u \Delta \varphi \, dx = \int_\Omega (\Delta_k u) \varphi \, dx \; \forall \varphi \in \mathring{H}^2(\Omega). \tag{6}$$

Airy stress function operator

The bilinear operator $\Phi : H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$:

$$\int_\Omega \Delta\Phi(u,v)\Delta\varphi\,dx = \int_\Omega [u,v]\varphi\,dx\ \forall\ \varphi \in \mathring{H}^2(\Omega). \tag{7}$$

The equation (7) has a unique solution, because $[u,v] \in L_1(\Omega) \hookrightarrow H^2(\Omega)^*$.

The operator Φ is compact and symmetric.
Moreover $\Phi : H^2(\Omega)^2 \to W^2_p(\Omega),\ 2 < p < \infty$ and

$$\|\Phi(u,v)\|_{W^2_p(\Omega)} \le c\|u\|_{H^2(\Omega)}\|v\|_{W^1_p(\Omega)}\ \forall u \in H^2(\Omega),\ v \in W^1_p(\Omega).$$

A new variational formulation

We reformulate the problem (4), (5) into

Problem $\mathcal{P}$. *Find $u \in \mathcal{C}$ such that*

$$\begin{aligned}
&\int_Q (E_1 A(\dot u, y-u) + E_0 A(u, y-u))\,dx\,dt + \\
&\int_Q [u, E_1\partial_t(\tfrac{1}{2}\Phi(u,u) + Lu) + E_0(\tfrac{1}{2}\Phi(u,u) + Lu)](y-u)\,dx\,dt \\
&+ \int_Q \left(\Delta_k\left(E_1\partial_t(\tfrac{1}{2}\Phi(u,u) + Lu) + E_0(\tfrac{1}{2}\Phi(u,u) + Lu)\right)\right) \\
&\qquad \times (y-u)\,dx\,dt \\
&- \int_Q (a\nabla\dot u\cdot\nabla(\dot y - \dot u) + \dot u(\dot y - \dot u))\,dx\,dt \\
&+ \int_\Omega (a\nabla\dot u\cdot\nabla(y-u) + \dot u(y-u))\,(T,\cdot)\,dx \\
&\geq \int_\Omega (a\nabla u_1\cdot\nabla(y(0,\cdot) - u_0) + u_1(y(0,\cdot) - u_0))\,dx \\
&+ \int_Q f(y-u)\,dx\,dt \ \forall y \in \mathcal{C}
\end{aligned}$$

Method of penalization

For any $\eta > 0$ we define *The Penalized Problem*

Problem $\mathcal{P}_\eta$. We look for $u \in H^{1,2}(Q)$ such that $\dot{u} \in L_2(I, H^2(\Omega))$ and

$$\int_Q \left(\ddot{u}z + a\nabla\ddot{u}\cdot\nabla z + E_1 A(\dot{u}, z) + E_0 A(u, z)\right) dx\, dt +$$

$$\begin{aligned} &\int_Q \Big([u, E_1\partial_t(\tfrac{1}{2}\Phi(u, u) + Lu) + E_0(\tfrac{1}{2}\Phi(u, u) + Lu)] + \\ &\Delta_k \left(E_1\partial_t(\tfrac{1}{2}\Phi(u, u) + Lu) + E_0(\tfrac{1}{2}\Phi(u, u) + Lu)\right)\Big) z\, dx\, dt \end{aligned} \tag{8}$$

$$= \int_Q (f + \eta^{-1}u^-) z\, dx\, dt \ \forall z \in L_2(I; H^2(\Omega)),$$

$$u(0, \cdot) = u_0 \geq 0,\ \dot{u}(0, \cdot) = u_1 \text{ on } \Omega, \tag{9}$$

with $u^- = \max\{0, -u\}$ a negative part of u.

Solving the penalized problem

The existence of a solution to the penalized problem using Galerkin method:

Lemma. *Let $f \in L_2(Q)$, $u_0 \in H^2(\Omega)$, and $u_1 \in H^1(\Omega)$. Then there exists a solution u_η of the problem $\mathcal{P}_\eta$* fulfilling the η-independent estimates

$$\|\dot{u}_\eta\|^2_{L_2(I;H^2(\Omega))} + \|\dot{u}_\eta\|^2_{L_\infty(I;H^1(\Omega))} + \|u_\eta\|^2_{L_\infty(I;H^2(\Omega))} + \|\partial_t\Phi(u_\eta, u_\eta)\|^2_{L_2(I;H^2(\Omega))} \le c, \tag{10}$$

$$\|\partial_t\Phi(u_\eta, u_\eta)\|_{L_2(I;W^2_p(\Omega))} \le c_p \; \forall\, p > 2.$$

Crucial estimates

The test function $z \equiv 1$ in $\mathcal{P}_\eta$ implies

$$\int_Q \eta^{-1} u_\eta^- \, dx\, dt = \int_Q (\ddot{u}_\eta - f)\, dx\, dt \\ = \int_\Omega (\dot{u}_\eta(T,\cdot) - u_1)\, dx - \int_Q f\, dx\, dt$$

and the L_1 estimates

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \leq c(f, u_0, u_1), \tag{11}$$

$$\|\dot{\psi}_\eta\|_{L_1(I;H^2(\Omega)^*)} \leq c(f, u_0, u_1), \tag{12}$$

where

$$\psi_\eta : w \mapsto \int_Q (a\nabla \dot{u}_\eta \nabla w + \dot{u}_\eta w)\, dx\, dt.$$

The convergence of the penalized sequence

The generalization of the Aubin's compactness lemma:
Lemma *Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ be Banach spaces, the first reflexive and separable. Let $1 < p < \infty,\ 1 \le q < \infty$. Then $W \equiv \{v;\ v \in L_p(I; B_0),\ \dot{v} \in L_q(I, B_1)\} \hookrightarrow\hookrightarrow L_p(I; B)$.*

There exists a sequence $\eta_k \searrow 0$ such that for $u_k \equiv u_{\eta_k}$ the following convergence hold for any real $p \geq 1$:

$$\begin{aligned} &\dot{u}_k \rightharpoonup \dot{u} \text{ in } L_2(I; H^2(\Omega)) \\ &\dot{u}_k \to \dot{u} \text{ in } L_2(I; W^1_p(\Omega)), \\ &u_k \to u \ \text{ in } C(I; W^1_p(\Omega)), \end{aligned} \tag{13}$$

Theorem. Let $u_0 \in H^2(\Omega),\ u_1 \in H^1(\Omega)$ and $f \in L_2(Q)$. Then there exists a solution u of the contact Problem $\mathcal{P}$.

Another boundary conditions

Remark. *The presented method can be applied also to more natural boundary conditions*

$$u = \partial_{\boldsymbol{n}} u = 0 \text{ on } \mathcal{S}$$

or

$$u = \mathcal{M}(u) = 0 \text{ on } \mathcal{S}$$

for clamped or simply supported shell.

We can assume also the unilateral condition

$$u(t,x) \geq \Psi(x) \text{ for a.e. } (t,x) \in Q.$$

Thank You