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OPTIMAL CONTROL FOR A CLASS OF HISTORY-DEPENDENT HEMIVARIATIONAL INEQUALITIES

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Abstract

We consider a class of optimal control problems for abstract evolution hemivariational inequalities. The latter are hyperbolic partial differential equations which appear in the study of frictional contact problems for viscoelastic materials with long memory term. In our model a dynamic equation of motion is considered with the viscoelastic constitutive relationship of the Kelvin-Voigt type and the contact is bilateral. The term responsible for memory of the body is described by a history-dependent operator which, at any moment $t \in (0,T)$ depends on the history of the solution up to the moment t. Moreover, the multivalued boundary conditions come from the nonconvex superpotentials and they can be written in the form of the Clarke subdifferentials for locally Lipschitz functions. This leads to the history-dependent subdifferential inclusion of the form (1).

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ and $V \subset Z \subset H \subset Z^* \subset V^*$ be an evolution fivefold with the compact embedding $V \subset Z$, and $\gamma \in \mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^d))$. Given $T < \infty$ we introduce the function spaces

$$\mathcal{V} = L^2(0,T;V), \quad \mathcal{Z} = L^2(0,T;Z), \quad \widehat{\mathcal{H}} = L^2(0,T;H) \ \text{ and } \ \mathcal{W} = \{v \in \mathcal{V} : v' \in \mathcal{V}^*\},$$

where the time derivative is understood in the sense of vector valued distributions. Let $A\colon (0,T)\times V \to V^*$, $B \in \mathcal{L}(V,V^*)$, $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$, $J\colon (0,T)\times L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ and $f \in \mathcal{V}^*$ be given. Then, we consider the following subdifferential problem.

Find $y \in \mathcal{V}$ such that $y' \in \mathcal{W}$ and

$$\begin{cases} y''(t) + A(t, y'(t)) + By(t) + \mathcal{S}y(t) + \gamma^* \partial J(t, \gamma y(t)) \ni f(t) \ \text{ a.e. } t \in (0,T) \\ y(0) = y_0, \ y'(0) = y_1. \end{cases} \tag{1}$$

Here $\mathcal{S}$ is the history-dependent operator; in particular case it can be given in the integral form of a linear continuous operator, i.e. the Volterra-type operator

$$\mathcal{S}v(t) = \int_0^t C(t-s)\, v(s)\, ds \quad \text{ for all } \ v \in \mathcal{V}, \ t \in (0,T),$$

with $C \in L^\infty(0,T;\mathcal{L}(V,V^*))$ (cf. [3]).

The aim of this paper is twofold. First we review recent results on the existence and uniqueness of weak solutions to dynamic history-dependent hemivariational inequalities. These results are based on the solvability of associated nonlinear evolution inclusions with pseudomonotone multivalued operators considered on Sobolev spaces of vector valued functions (cf. [2]).

In the second part we study the optimal control problem for hemivariational inclusion of the form

$$\begin{cases} y''(t) + A(t, y'(t)) + By(t) + \mathcal{S}y(t) + \gamma^* \partial J(t, \gamma y'(t)) \ni f(t) + E(t)u(t) \quad \text{a.e. } t \\ y(0) = y_0, \ y'(0) = y_1, \end{cases} \tag{2}$$

where $u \in \mathcal{U} = L^2(0, T; U)$ is a control, U is a space of control variables and $E \in L^\infty(0, T; \mathcal{L}(U, Z))$ represents a controller.

We consider the Boltza distributed parameter control problem

$$\begin{cases} \Phi(y, u) = l(y(T), y'(T)) + \displaystyle\int_0^T F(t, y(t), y'(t), u(t))\, dt \longrightarrow \inf \\ \text{with } y = y(u) - \text{a solution state of (2) corresponding to } u, \end{cases} \tag{3}$$

where $l \colon H \times H \to \mathbb{R}$, $F \colon (0, T) \times H \times H \times U \to \mathbb{R} \cup \{+\infty\}$ are given functions.

We deliver results on the continuous dependence of the solution to hemivariational inequality on the control parameter. We provide conditions that guarantee the existence of optimal solutions to control problems by applying the direct method of the calculus of variations.

Finally, we provide a mechanical model and examples of nonsmooth and nonconvex superpotentials to which the theory applies (cf. [2]). The physical setting is the following. A viscoelastic body occupies a regular domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ with surface Γ divided into three disjoint parts Γ_D, Γ_N and Γ_C, such that $\operatorname{meas}(\Gamma_D) > 0$. We assume that the body is clamped on Γ_D, so the displacement field vanishes there. Volume forces of density $\boldsymbol{f}_0$ act in Ω and surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. The body may come in contact with an obstacle over the potential contact surface Γ_C. The contact is frictional and is modeled with subdifferential boundary conditions. We use the notation $Q = \Omega \times (0, T)$ and $\mathbb{S}^d = \mathbb{R}_s^{d \times d}$ for the space of symmetric matrices of order d. Then the frictional contact problem under consideration can be stated sa follows.

Find the displacement $\boldsymbol{y} : Q \to \mathbb{R}^d$ and the stress field $\boldsymbol{\sigma} : Q \to \mathbb{S}^d$ such that

$$\begin{cases} \boldsymbol{y}''(t) - \operatorname{Div} \boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) & \text{in } Q, \\ \boldsymbol{\sigma}(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}'(t))) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{y}(t)) + \displaystyle\int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{y}(s))\, ds & \text{in } Q, \\ \boldsymbol{y}(t) = \boldsymbol{0} & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) & \text{on } \Gamma_N, \\ -\sigma_\nu(t) \in \partial j_\nu(t, y'_\nu(t)) & \text{on } \Gamma_C, \\ -\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(t, \boldsymbol{y}'_\tau(t)) & \text{on } \Gamma_C, \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \ \boldsymbol{y}'(0) = \boldsymbol{y}_1 & \text{in } \Omega, \end{cases} \tag{4}$$

where j_ν and j_τ are given superpotentials, $\boldsymbol{\varepsilon}(\boldsymbol{y})$ denotes the linearized strain tensor and y'_ν, $\boldsymbol{y}'_\tau$, σ_ν, $\boldsymbol{\sigma}_\tau$ are normal and tangential components of velocity vectors and stress tensors, respectively. Here $\mathcal{A}$ is a nonlinear operator describing the viscous properties of the material while $\mathcal{B}$, $\mathcal{C}$ are the linear elasticity and relaxation operators. The symbol ∂j denotes the Clarke subdifferential of j with respect to the last variable. Note also that the explicit dependence of the functions j_n and j_τ as well as of the operator $\mathcal{A}$ with respect to the time variable allows to model situations when the frictional contact conditions and the viscosity properties of the material depend on the temperature, which plays the role of a parameter, and whose evolution in time is prescribed.

The variational formulation of the mechanical problem (4) leads to the history-dependent hemivariational inequality which corresponds to the inclusion (1).

Concrete examples of frictional models which lead to these boundary conditions include the viscous contact and the contact with nonmonotone normal damped response, associated to a nonmonotone friction law, to Tresca's friction law or to a power-law friction (cf. [1]).

References

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