(1) Riešte okrajovú úlohu

$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ 0 < x < \pi, \ 0 < y < 2,$
$u(0,y) = u(\pi,y) = 0 = \frac{\partial u}{\partial y}(x,0) = 0, \ \frac{\partial u}{\partial y}(x,2) = x.$

$u(x,y) = X(x)Y(y),$
$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda,$
$Y''(y) - \lambda Y(y) = 0.$
$X''(x) + \lambda X(x) = 0, \ X(0) = X(\pi) = 0,$
$\lambda > 0, \ X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$

$X(0) = c_1 = 0, \ c_2 = 1,$

$X(\pi) = \sin\sqrt{\lambda}\pi = 0,$
$\sqrt{\lambda}\pi = n\pi, \ \sqrt{\lambda} = n, \ n \in N,$
$\lambda \equiv \lambda_n = n^2, \ X(x) \equiv X_n(x) = \sin nx, \ n \in N.$

$Y_n''(y) - n^2 Y_n(y) = 0, \ n \in N,$
$Y_n(y) = a_n \cosh ny + b_n \sinh ny.$
$Y_n'(y) = na_n \sinh ny + nb_n \cosh ny.$
$Y_n'(0) = nb_n = 0, \ \Rightarrow \ Y_n(y) = a_n \cosh ny, \ n \in N.$

$u_n(x,y) = Y_n(y)X_n(x) = a_n \cosh ny \sin nx,$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} a_n \cosh ny \sin nx.$$

$x = \frac{\partial u}{\partial y}(x,2) = \sum_{n=1}^{\infty} na_n \sinh 2n \sin nx.$
$A_n = na_n \sinh 2n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx =$
$\frac{2}{\pi}\left\{\left[x\frac{-\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right\} = \frac{2(-1)^{n-1}}{n}.$

$$a_n = \frac{A_n}{n \sinh 2n} = \frac{2(-1)^{n-1}}{n^2 \sinh 2n}, \ n \in N.$$

$$\underline{u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^2 \sinh 2n} \cosh ny \sin nx.}$$

(2) Riešte okrajovú úlohu

$\Delta u(x,y) = 0,\ x^2+y^2<9,\ x>0,\ y>0,$

$\frac{\partial u}{\partial y}(x,0) = u(0,y) = 0,\ \frac{\partial u}{\partial \vec{n}}(x,y) = 1,$ ak $x^2+y^2=9$

$r^2\Delta u(r,\varphi) = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0,\ 0<r<3,\ 0<\varphi<\frac{\pi}{2}.$

$u(r,\varphi) = R(r)\Phi(\varphi),$

$\frac{r^2R''(r)+rR'(r)}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda$

$r^2R''+rR'-\lambda R = 0,\ 0<r<3.$

$\Phi''+\lambda\Phi = 0,\ 0<\varphi<\frac{\pi}{2};\ \Phi'(0) = \Phi(\frac{\pi}{2}) = 0.$

$\lambda>0,\ \Phi(\varphi) = c_1\cos(\sqrt{\lambda}\varphi) + c_2\sin(\sqrt{\lambda}\varphi),$

$\Phi'(0) = \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0,\ c_1 = 1$

$\Phi(\frac{\pi}{2}) = \cos(\sqrt{\lambda}\frac{\pi}{2}) = 0 \Rightarrow \sqrt{\lambda}\frac{\pi}{2} = (2n-1)\frac{\pi}{2},\ n\in N$

$\lambda \equiv \lambda_n = (2n-1)^2,\ \Phi_n(\varphi) = \cos(2n-1)\varphi,\ n\in N$

$r^2R_n''+rR_n'-(2n-1)^2R_n = 0,\ 0<r<3;$

$R_n(r) = a_nr^{2n-1},\ n\in N$

$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi),\ n\in N$

$u(r,\varphi) = \sum_{n=1}^{\infty} R_n(r)\Phi_n(\varphi) = \sum_{n=1}^{\infty} a_nr^{2n-1}\cos(2n-1)\varphi,$

$1\ =\ \frac{\partial u}{\partial r}(3,\varphi) = \sum_{n=1}^{\infty} na_n(2n-1)3^{2n-2}\cos(2n-1)\varphi,$

$A_n = (2n-1)3^{2n-2}a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos(2n-1)\varphi\,d\varphi = \frac{4(-1)^{n-1}}{(2n-1)\pi},$

$$a_n = \frac{A_n}{(2n-1)3^{2n-2}} = \frac{4(-1)^{n-1}}{(2n-1)^2\pi 3^{2n-2}},\ n\in N.$$

$$\underline{u(r,\varphi) = \sum_{n=1}^{\infty}\frac{12(-1)^{n-1}}{(2n-1)^2\pi}\left(\frac{r}{3}\right)^{2n-1}\cos(2n-1)\varphi.}$$