

(1) Riešte okrajovú úlohu

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < 2,$$

$$u(0, y) = u(\pi, y) = 0 = \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 2) = x.$$

$$u(x, y) = X(x)Y(y),$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

$$Y''(y) - \lambda Y(y) = 0.$$

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0,$$

$$\lambda > 0, \quad X(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x,$$

$$X(0) = c_1 = 0, \quad c_2 = 1,$$

$$X(\pi) = \sin \sqrt{\lambda}\pi = 0,$$

$$\sqrt{\lambda}\pi = n\pi, \quad \sqrt{\lambda} = n, \quad n \in N,$$

$$\lambda \equiv \lambda_n = n^2, \quad X(x) \equiv X_n(x) = \sin nx, \quad n \in N.$$

$$Y_n''(y) - n^2 Y_n(y) = 0, \quad n \in N,$$

$$Y_n(y) = a_n \cosh ny + b_n \sinh ny.$$

$$Y_n'(y) = na_n \sinh ny + nb_n \cosh ny.$$

$$Y_n'(0) = nb_n = 0, \quad \Rightarrow \quad Y_n(y) = a_n \cosh ny, \quad n \in N.$$

$$u_n(x, y) = Y_n(y)X_n(x) = a_n \cosh ny \sin nx,$$

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} a_n \cosh ny \sin nx.$$

$$x = \frac{\partial u}{\partial y}(x, 2) = \sum_{n=1}^{\infty} na_n \sinh 2n \sin nx.$$

$$A_n = na_n \sinh 2n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx =$$

$$\frac{2}{\pi} \left\{ \left[x \frac{-\cos nx}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \, dx \right\} = \frac{2(-1)^{n-1}}{n}.$$

$$a_n = \frac{A_n}{n \sinh 2n} = \frac{2(-1)^{n-1}}{n^2 \sinh 2n}, \quad n \in N.$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^2 \sinh 2n} \cosh ny \sin nx.$$

(2) Riešte okrajovú úlohu

$$\Delta u(x, y) = 0, \quad x^2 + y^2 < 9, \quad x > 0, \quad y > 0, \\ \frac{\partial u}{\partial y}(x, 0) = u(0, y) = 0, \quad \frac{\partial u}{\partial n}(x, y) = 1, \quad \text{ak } x^2 + y^2 = 9$$

$$r^2 \Delta u(r, \varphi) = \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 < r < 3, \quad 0 < \varphi < \frac{\pi}{2}.$$

$$u(r, \varphi) = R(r)\Phi(\varphi),$$

$$\frac{r^2 R'' + r R'}{R(r)} = -\frac{\Phi''}{\Phi(\varphi)} = \lambda$$

$$r^2 R'' + r R' - \lambda R = 0, \quad 0 < r < 3.$$

$$\Phi'' + \lambda \Phi = 0, \quad 0 < \varphi < \frac{\pi}{2}; \quad \Phi'(0) = \Phi\left(\frac{\pi}{2}\right) = 0.$$

$$\lambda > 0, \quad \Phi(\varphi) = c_1 \cos(\sqrt{\lambda} \varphi) + c_2 \sin(\sqrt{\lambda} \varphi),$$

$$\Phi'(0) = \sqrt{\lambda} c_2 = 0 \Rightarrow c_2 = 0, \quad c_1 = 1$$

$$\Phi\left(\frac{\pi}{2}\right) = \cos(\sqrt{\lambda} \frac{\pi}{2}) = 0 \Rightarrow \sqrt{\lambda} \frac{\pi}{2} = (2n-1) \frac{\pi}{2}, \quad n \in N$$

$$\lambda \equiv \lambda_n = (2n-1)^2, \quad \Phi_n(\varphi) = \cos(2n-1)\varphi, \quad n \in N$$

$$r^2 R_n'' + r R_n' - (2n-1)^2 R_n = 0, \quad 0 \leq r < 3;$$

$$R_n(r) = a_n r^{2n-1}, \quad n \in N$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} a_n r^{2n-1} \cos(2n-1)\varphi,$$

$$1 = \frac{\partial u}{\partial r}(3, \varphi) = \sum_{n=1}^{\infty} n a_n (2n-1) 3^{2n-2} \cos(2n-1)\varphi,$$

$$A_n = (2n-1) 3^{2n-2} a_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos(2n-1)\varphi \, d\varphi = \frac{4(-1)^{n-1}}{(2n-1)\pi},$$

$$a_n = \frac{A_n}{(2n-1) 3^{2n-2}} = \frac{4(-1)^{n-1}}{(2n-1)^2 \pi 3^{2n-2}}, \quad n \in N.$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{12(-1)^{n-1}}{(2n-1)^2 \pi} \left(\frac{r}{3}\right)^{2n-1} \cos(2n-1)\varphi.$$
