

(1) Riešte okrajovú úlohu

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 1,$$

$$\frac{\partial u}{\partial y}(x, 0) = u(x, 1) = u(0, y) = 0, \quad \frac{\partial u}{\partial x}(2, y) = 1.$$

$$u(x, y) = X(x)Y(y),$$

$$X''(x) - \lambda X(x) = 0$$

$$Y''(y) + \lambda Y(y) = 0, \quad Y'(0) = Y(1) = 0.$$

$$\lambda > 0, \quad Y(y) = c_1 \cos \sqrt{\lambda} y + c_2 \sin \sqrt{\lambda} y,$$

$$Y'(y) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} y + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} y,$$

$$Y'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0, \quad c_1 = 1,$$

$$Y(y) = \cos \sqrt{\lambda} y, \quad Y(1) = \cos \sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = \frac{(2k-1)\pi}{2},$$

$$\lambda \equiv \lambda_k = \frac{(2k-1)^2 \pi^2}{4}, \quad Y(y) \equiv Y_k(y) = \cos \frac{(2k-1)\pi}{2} y, \quad k = 1, 2, \dots$$

$$X_k'' - \lambda_k X_k(x) = 0, \quad X_k'' - \frac{(2k-1)^2 \pi^2}{4} X_k(x) = 0$$

$$X_k(x) = a_k \cosh \frac{(2k-1)\pi}{2} x + b_k \sinh \frac{(2k-1)\pi}{2} x.$$

$$X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh \frac{(2k-1)\pi}{2} x.$$

$$u_k(x, y) = X_k(x)Y_k(y) = b_k \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y,$$

$$u(x, y) = \sum_{k=1}^{\infty} u_k(x, y) = \sum_{k=1}^{\infty} b_k \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y.$$

$$1 = \frac{\partial u}{\partial y}(2, y) = \sum_{k=1}^{\infty} b_k \frac{(2k-1)\pi}{2} \cosh(2k-1)\pi \cos \frac{(2k-1)\pi}{2} y.$$

$$B_k = b_k \frac{(2k-1)\pi}{2} \cosh(2k-1)\pi = 2 \int_0^1 1 \cos \frac{(2k-1)\pi}{2} y dy$$

$$= \frac{4}{(2k-1)\pi} \sin \frac{(2k-1)\pi}{2} = \frac{4(-1)^{k-1}}{(2k-1)\pi}.$$

$$b_k = \frac{B_k}{\frac{(2k-1)\pi}{2} \cosh(2k-1)\pi} = \frac{8(-1)^{k-1}}{(2k-1)^2 \pi^2 \cosh(2k-1)\pi}.$$

$$u(x, y) = \sum_{k=1}^{\infty} \frac{8(-1)^{k-1}}{(2k-1)^2 \pi^2 \cosh(2k-1)\pi} \sinh \frac{(2k-1)\pi}{2} x \cos \frac{(2k-1)\pi}{2} y.$$

(2) Riešte okrajovú úlohu

$$\Delta u(x, y) = 1, \quad x^2 + y^2 < 1, \quad u(x, y) = 0, \quad \text{ak } x^2 + y^2 = 1.$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 1, \quad 0 < r < 1.$$

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = r^2, \quad 0 \leq r < 1, \quad 0 \leq \phi \leq 2\pi, \quad u(1, \phi) = 0$$

$$u(r, \phi) = R(r), \quad 0 \leq r < 1, \quad R(1) = 0.$$

$$r^2 R''(r) + r R'(r) = r^2, \quad r R'' + R' = r,$$

$$(r R'(r))' = r, \quad R(1) = 0,$$

$$r R'(r) = \frac{1}{2} r^2 + C, \quad R'(r) = \frac{1}{2} r + \frac{C_1}{r},$$

$$R(r) = \frac{1}{4} r^2 + C_1 \ln r + C_2, \quad C_1 = 0, \text{ pretože riešenie je ohraničené.}$$

$$R(1) = \frac{1}{4} + C_2 = 0 \Rightarrow C_2 = -\frac{1}{4},$$

$$R(r) = u(r, \phi) = \frac{1}{4} r^2 - \frac{1}{4} = \frac{1}{4} (r^2 - 1),$$

$$u(x, y) = \frac{1}{4} (x^2 + y^2 - 1).$$

(3) Riešte začiatočno-okrajovú úlohu

$$\begin{aligned}
& \frac{\partial u}{\partial t} - 4 \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < \pi, \\
& u(0, x) = x, \quad u(t, 0) = u(t, \pi) = 0. \\
& u(t, x) = T(t)X(x), \quad \frac{T'(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\lambda \\
& T'(t) + 4\lambda T(t) = 0, \quad t > 0, \\
& X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0, \quad \lambda > 0. \\
& X(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x, \\
& X(0) = c_1 = 0, \quad c_2 = 1, \quad X(x) = \sin \sqrt{\lambda}x, \\
& X(\pi) = \sin \sqrt{\lambda}\pi = 0, \Rightarrow \sqrt{\lambda}\pi = k\pi, \\
& \lambda \equiv \lambda_k = k^2, \quad X \equiv X_k(x) = \sin kx, \quad k = 1, 2, \dots \\
& T'_k(t) + 4\lambda_k T_k(t) = T'_k(t) + 4k^2 T_k(t) = 0, \quad t > 0, \\
& T_k(t) = a_k e^{-4k^2 t}, \quad t > 0, \\
& u_k(t, x) = T_k(t)X_k(x) = a_k e^{-4k^2 t} \sin kx, \\
& u(t, x) = \sum_{k=1}^{\infty} u_k(x, y) = \sum_{k=1}^{\infty} a_k e^{-4k^2 t} \sin kx, \\
& x = u(0, x) = \sum_{k=1}^{\infty} a_k \sin kx, \\
& a_k = \frac{2}{\pi} \int_0^{\pi} x \sin kx \, dx = \frac{2}{\pi} \left\{ \left[x \frac{(-\cos kx)}{k} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos kx}{k} \, dx \right\} = \frac{2(-1)^{k-1}}{k}. \\
& \underline{u(t, x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{k} e^{-4k^2 t} \sin kx.}
\end{aligned}$$

(4) Riešte okrajovú úlohu

$$\begin{aligned}
& \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < \pi, \\
& \frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, \pi) = u(0, y) = 0, \quad u(2, y) = 1. \\
& u(x, y) = X(x)Y(y), \\
& X''(x) - \lambda X(x) = 0 \\
& Y''(y) + \lambda Y(y) = 0, \quad Y'(0) = Y'(\pi) = 0. \\
& \text{a) } \lambda_0 = 0, \quad Y''(y) = 0 \Rightarrow Y_0(y) = c_1 y + c_2. \\
& Y'_0(0) = Y'_0(\pi) = c_1 = 0, \quad c_2 = 1 \Rightarrow Y_0(y) = 1. \\
& \text{b) } \lambda > 0, \quad Y(y) = c_1 \cos \sqrt{\lambda}y + c_2 \sin \sqrt{\lambda}y, \\
& Y'(y) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda}y + c_2 \sqrt{\lambda} \cos \sqrt{\lambda}y, \\
& Y'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0, \quad c_1 = 1, \\
& Y(y) = \cos \sqrt{\lambda}y, \quad Y'(\pi) = -\sin \sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = k\pi, \\
& \lambda \equiv \lambda_k = k^2, \quad Y(y) \equiv Y_k(y) = \cos ky, \quad k = 0, 1, 2, \dots \\
& X''_k - \lambda_k X_k(x) = 0, \quad X''_k - k^2 X_k(x) = 0, \quad k = 0, 1, 2, \dots \\
& X_0(x) = a_0 + b_0 x, \quad X_0(0) = a_0 = 0 \Rightarrow X_0(x) = b_0 x, \\
& X_k(x) = a_k \cosh kx + b_k \sinh kx. \\
& X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh kx, \quad k = 1, 2, \dots
\end{aligned}$$

$$\begin{aligned}
u_0(x, y) &= X_0(x)Y_0(y) = b_0x, \\
u_k(x, y) &= X_k(x)Y_k(y) = b_k \sinh kx \cos ky, \quad k = 1, 2, \dots \\
u(x, y) &= \sum_{k=0}^{\infty} u_k(x, y) = b_0x + \sum_{k=1}^{\infty} b_k \sinh kx \cos ky. \\
1 &= u(2, y) = 2b_0 + \sum_{k=1}^{\infty} b_k \sinh 2k \cos ky. \\
B_0 &= 2b_0 = \frac{\int_0^{\pi} 1^2 dy}{\int_0^{\pi} 1^2 dy} = 1 \Rightarrow b_0 = \frac{1}{2}, \\
B_k &= b_k \sinh 2k = \frac{2}{\pi} \int_0^{\pi} 1 \cos ky dy = 0 \Rightarrow b_k = 0, \quad k = 1, 2, \dots \\
\hline
u(x, y) &= \frac{1}{2}x.
\end{aligned}$$

(5) Riešte okrajovú úlohu

$$\begin{aligned}
\Delta u(x, y) &= 0, \quad x^2 + y^2 < 1, \quad y > 0, \quad u(x, y) = 1, \quad \text{ak } x^2 + y^2 = 1. \\
r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} &= 0, \quad 0 \leq r < 1, \quad 0 < \phi < \pi, \\
u(r, 0) &= u(r, \pi) = 0, \quad u(1, \phi) = 1. \\
u(r, \phi) &= R(r)\Phi(\phi), \\
\frac{r^2 R''(r) + rR'(r)}{R(r)} &= -\frac{\Phi''(\phi)}{\Phi(\phi)} = \lambda. \\
r^2 R''(r) + rR'(r) - \lambda R(r) &= 0, \quad 0 \leq r < 1, \\
\Phi''(\phi) + \lambda \Phi(\phi) &= 0, \quad 0 < \phi < \pi, \quad \Phi(0) = \Phi(\pi) = 0, \\
\lambda > 0, \quad \Phi(\phi) &= c_1 \cos \sqrt{\lambda} \phi + c_2 \sin \sqrt{\lambda} \phi, \\
\Phi(0) = c_1 &= 0, \quad c_2 = 1, \rightarrow \Phi(\phi) = \sin \sqrt{\lambda} \phi, \\
\Phi(\pi) = \sin \sqrt{\lambda} \pi &= 0, \Rightarrow \sqrt{\lambda} \pi = k\pi, \\
\lambda \equiv \lambda_k &= k^2, \quad \Phi(\phi) \equiv \Phi_k(\phi) = \sin k\phi, \\
r^2 R''(r) + rR'(r) - k^2 R(r) &= 0, \quad 0 \leq r < 1, \\
R(r) &= a_k r^k + b_k r^{-k}, \quad k = 1, 2, \dots \\
b_k &= 0, \quad \text{pretože riešenie je ohraničené.} \\
R(r) \equiv R_k(r) &= a_k r^k, \quad k = 1, 2, \dots, \quad 0 < r < 1, \\
u_k(r, \phi) &= R_k(r)\Phi_k(\phi) = a_k r^k \sin k\phi, \quad k = 1, 2, \dots \\
u(r, \phi) &= \sum_{k=1}^{\infty} u_k(r, \phi) = \sum_{k=1}^{\infty} a_k r^k \sin k\phi. \\
1 &= u(1, \phi) = \sum_{k=1}^{\infty} a_k \sin k\phi. \\
a_k &= \frac{2}{\pi} \int_0^{\pi} \sin k\phi d\phi = \frac{2}{\pi} \left[\frac{-\cos k\phi}{k} \right]_0^{\pi} = \frac{2}{k\pi} [1 - (-1)^k], \\
\hline
u(r, \phi) &= \sum_{k=1}^{\infty} \frac{2}{k\pi} [1 - (-1)^k] r^k \sin k\phi.
\end{aligned}$$

(6) Riešte začiatovo-okrajovú úlohu

$$\begin{aligned}
\frac{\partial u}{\partial t} - 9 \frac{\partial^2 u}{\partial x^2} &= 0, \quad t > 0, \quad 0 < x < 2, \\
u(0, x) &= x, \quad u(t, 0) = u(t, 2) = 0. \\
u(t, x) &= T(t)X(x), \quad \frac{T'(t)}{9T(t)} = \frac{X''(x)}{X(x)} = -\lambda \\
T'(t) + 9\lambda T(t) &= 0, \quad t > 0, \\
X''(x) + \lambda X(x) &= 0, \quad X(0) = X(2) = 0, \quad \lambda > 0. \\
X(x) &= c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x, \\
X(0) = c_1 &= 0, \quad c_2 = 1, \quad X(x) = \sin \sqrt{\lambda} x,
\end{aligned}$$

$$X(2) = \sin \sqrt{\lambda} 2 = 0 \Rightarrow 2\sqrt{\lambda} = k\pi,$$

$$\lambda \equiv \lambda_k = \frac{k^2\pi^2}{4}, \quad X \equiv X_k(x) = \sin \frac{k\pi}{2}x, \quad k = 1, 2, \dots$$

$$T'_k(t) + 9\lambda_k T_k(t) = T'_k(t) + 9\frac{k^2\pi^2}{4}T_k(t) = 0, \quad t > 0,$$

$$T_k(t) = a_k e^{-\frac{9k^2\pi^2}{4}t}, \quad t > 0,$$

$$u_k(t, x) = T_k(t)X_k(x) = a_k e^{-\frac{9k^2\pi^2}{4}t} \sin \frac{k\pi}{2}x,$$

$$u(t, x) = \sum_{k=1}^{\infty} u_k(x, y) = \sum_{k=1}^{\infty} a_k e^{-\frac{9k^2\pi^2}{4}t} \sin \frac{k\pi}{2}x,$$

$$x = u(0, x) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi}{2}x,$$

$$a_k = \int_0^2 x \sin \frac{k\pi}{2}x \, dx =$$

$$\left[x \left(-\frac{2}{k\pi} \right) \cos \frac{k\pi}{2}x \right]_0^2 + \int_0^2 \frac{2}{k\pi} \cos \frac{k\pi}{2}x \, dx = \frac{4(-1)^{k-1}}{k\pi}.$$

$$u(t, x) = \sum_{k=1}^{\infty} \frac{4(-1)^{k-1}}{k\pi} e^{-\frac{9k^2\pi^2}{4}t} \sin \frac{k\pi}{2}x, \quad t > 0, \quad 0 < x < 2.$$
