

(1) Riešte okrajovú úlohu $-(1+x^2)u'' - 2xu = 2$, $u(0) = u'(1) = 0$

$$-((1+x^2)u')' = 2, \quad ((1+x^2)u')' = -2,$$

$$(1+x^2)u' = -2x + c_1, \quad u' = \frac{-2x+c_1}{1+x^2}$$

$$u(x) = \int \frac{-2x+c_1}{1+x^2} dx = -\ln(1+x^2) + c_1 \arctan x + c_2,$$

$$u(0) = c_2 = 0, \quad u'(1) = \frac{-2+c_1}{2} = 0 \Rightarrow c_1 = 2,$$

$$u(x) = 2 \arctan x - \ln(1+x^2), \quad 0 < x < 1.$$

(2) Riešte okrajovú úlohu $-u'' + 4u = 0$, $u(0) = 1$, $u(2) + u'(2) = 0$

$$u(x) = c_1 \cosh 2x + c_2 \sinh 2x.$$

$$u(0) = c_1 = 1,$$

$$u(2) + u'(2) = \cosh 4 + c_2 \sinh 4 + 2 \sinh 4 + c_2 \cosh 4 = 0,$$

$$c_2 = -\frac{\cosh 4 + 2 \sinh 4}{\sinh 4 + 2 \cosh 4},$$

$$u(x) = \cosh 2x - \frac{\cosh 4 + 2 \sinh 4}{\sinh 4 + 2 \cosh 4} \sinh 2x$$

$$= \frac{1}{3e^4 + e^{-4}} (e^{2x-4} + 3e^{4-2x}).$$

(3) Riešte Sturmovu-Liovilleovu úlohu $u'' + \lambda u = 0$, $0 < x < 2\pi$,

a) $u'(0) = u'(2\pi) = 0$.

$$\lambda \geq 0.$$

$$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u(x) = c_1 x + c_2.$$

$$u'(0) = u'(2\pi) = c_1 = 0, \quad c_2 = 1,$$

$$\lambda_0 = 0, \quad u_0(x) = 1,$$

$$\lambda > 0 \Rightarrow u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

$$u'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x,$$

$$u'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0, \quad c_1 = 1$$

$$u'(2\pi) = \sqrt{\lambda} \sin \sqrt{\lambda} 2\pi = 0 \Rightarrow \sqrt{\lambda} 2\pi = n\pi, \quad n = 1, 2, \dots$$

$$\lambda_n = \frac{n^2}{4}, \quad u_n(x) = \cos \frac{n}{2} x, \quad n = 0, 1, 2, \dots$$

b) $u(0) = u'(2\pi) = 0$.

$$\lambda > 0 \Rightarrow u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

$$u'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x,$$

$$u(0) = c_1 = 0, \quad c_2 = 1$$

$$u'(2\pi) = \sqrt{\lambda} \cos \sqrt{\lambda} 2\pi = 0 \Rightarrow \sqrt{\lambda} 2\pi = (2n-1)\frac{\pi}{2},$$

$$\lambda_n = \frac{(2n-1)^2}{16}, \quad u_n(x) = \sin \frac{2n-1}{4} x, \quad n = 1, 2, \dots$$