(1) Riešte okrajovú úlohu

$\Delta u(x,y) = 0,\ x^2+y^2 < 16,\ y > 0;$
$u(x,0) = 0,\ \frac{\partial u}{\partial \vec{n}} = \arctan\frac{y}{x},\ \text{ak } x^2+y^2 = 16.$

*Riešenie:*
$\Delta u(r,\varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \varphi} = 0,\ u(r,0) = u(r,\pi) = 0,\ \frac{\partial u}{\partial r}(4,\varphi) = \varphi.$
$r^2\Delta u(r,\varphi) = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial^2 \varphi} = 0,\ 0 \le r < 4,\ 0 < \varphi < \pi.$

Dosadiť $u(r,\varphi) = R(r)\Phi(\varphi)$ :
$r^2R''\Phi + rR'\Phi + R\Phi'' = 0/\frac{1}{R\Phi},$

$\frac{r^2R''+rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$

$r^2R'' + rR' - \lambda R(r) = 0,\ 0 \le r < 4,$

$\Phi'' + \lambda\Phi = 0,\ \Phi(0) = \Phi(\pi) = 0.$

Z prvej okrajovej podmienky: $\lambda > 0,\ \Phi(\varphi) = \sin\sqrt{\lambda}\varphi.$
Z druhej okrajovej podmienky:
$\Phi(\pi) = \sin\sqrt{\lambda}\pi = 0,\ \Rightarrow\ \sqrt{\lambda}\pi = n\pi,\ n = 1,2,...$
$\lambda \equiv \lambda_n = n^2,\ \Phi \equiv \Phi_n(\varphi) = \sin n\varphi.$

$r^2R_n'' + rR_n' - n^2R_n(r) = 0,\ 0 \le r < 4,$
$R_n(r) = a_nr^n + b_nr^{-n},\ b_n = 0$ - ohraničenosť riešenia v nule.
$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_nr^n\sin n\varphi.$

$u(r,\varphi) = \sum_{n=1}^{\infty} a_nr^n\sin n\varphi.$

Nenulová okrajová podmienka:
$\frac{\partial u}{\partial r}(4,\varphi) = \varphi = \sum_{n=1}^{\infty} a_nn4^{n-1}\sin n\varphi.$

Fourierove koeficienty:
$A_n = a_nn4^{n-1} = \frac{2}{\pi}\int_0^{\pi}\varphi\sin n\varphi\,d\varphi = \frac{2(-1)^{n-1}}{n}$ po metóde per partes.
$a_n = \frac{2(-1)^{n-1}}{n^24^{n-1}}.$

*Výsledok:* $u(r,\varphi) = \sum_{n=1}^{\infty}\frac{8(-1)^{n-1}}{n^2}(\frac{r}{4})^n\sin n\varphi.$

(2) Riešte okrajovú úlohu

$\Delta = 0,\ 0 < x < \pi,\ 0 < y < 2,$
$u(0, y) = \frac{\partial u}{\partial x}(\pi, y) = u(x, 0) = 0,\ u(x, 2) = 1.$

*Riešenie:*

$u(x, y) = X(x)Y(y).$
Po dosadení do rovnice:
$X''Y + XY'' = 0/\frac{1}{XY},$

$Y'' - \lambda Y(y) = 0,$
$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$

$X'' + \lambda X = 0,\ X(0) = X'(\pi) = 0.$
Z prvej okrajovej podmienky: $\lambda > 0,\ X(x) = \sin\sqrt{\lambda}x,$
Z druhej okrajovej podmienky:
$X'(\pi) = \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = (2n-1)\frac{\pi}{2}.$
$\lambda \equiv \lambda_n = (\frac{2n-1}{2})^2,\ X \equiv X_n(x) = \sin\frac{2n-1}{2}x.$

Dosadením do rovnice pre $Y$:
$Y_n'' - (\frac{2n-1}{2})^2 Y_n = 0,\ Y_n(0) = 0.$
$Y_n(y) = a_n \cosh\frac{2n-1}{2}y + b_n \sinh\frac{2n-1}{2}y,\ 0 < y < 2.$
Z okrajovej podmienky: $a_n = 0,\ Y_n(y) = b_n \sinh\frac{2n-1}{2}y.$

$u_n(x, y) = Y_n(y)X_n(x) = b_n \sinh\frac{2n-1}{2}y \sin\frac{2n-1}{2}x.$

$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh\frac{2n-1}{2}y \sin\frac{2n-1}{2}x.$

Z nenulovej okrajovej podmienky:
$u(x, 2) = 1 = \sum_{n=1}^{\infty} b_n \sinh(2n-1) \sin\frac{2n-1}{2}x.$

Fourierove koeficienty:
$B_n = b_n \sinh(2n-1) = \frac{2}{\pi}\int_0^{\pi} 1 \sin\frac{2n-1}{2}x = \frac{4}{\pi(2n-1)},$
$b_n = \frac{4}{\pi(2n-1)\sinh(2n-1)}.$

*Výsledok:*
$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)\sinh(2n-1)} \sinh\frac{2n-1}{2}y \sin\frac{2n-1}{2}x.$