(1) Riešte okrajovú úlohu

$u'' - 9u = 9x^2,\ u(0) - u'(0) = 0,\ u(2) = 0.$

$u(x) = c_1 \cosh 3x + c_2 \sinh 3x + u_p(x),\ u_p(x) = ax^2 + bx + c,$

$u_p(x)'' - 9u_p(x) = 9x^2,$

$2a - 9c - 9bxb - ax^2 = 9x^2,$

$$a = -1,\ b = 0,\ c = -\frac{2}{9}$$

$$u(x) = c_1 \cosh 3x + c_2 \sinh 3x - x^2 - \frac{2}{9}$$

$$u(0) - u'(0) = c_1 - 3c_2 - \frac{2}{9} = 0,$$

$$u(2) = c_1 \cosh 6 + c_2 \sinh 6 - \frac{38}{9} = 0.$$

$$(3\cosh 6 + \sinh 6)c_2 = \frac{38}{9} - \frac{2}{9}\cosh 6$$

$$c_2 = \frac{38 - 2\cosh 6}{9(3\cosh 6 + \sinh 6)}$$

$$c_1 = 3c_2 + \frac{2}{9} = \frac{38 - 2\cosh 6}{3(3\cosh 6 + \sinh 6)} + \frac{2}{9} = \frac{114 + 2\sinh 6}{9(3\cosh 6 + \sinh 6)}.$$

$$u(x) = \frac{114 + 2\sinh 6}{9(3\cosh 6 + \sinh 6)}\cosh 3x + \frac{38 - 2\cosh 6}{9(3\cosh 6 + \sinh 6)}\sinh 3x - x^2 - \frac{2}{9}$$

$$= \frac{152 + 2\sinh(6 - 3x)}{9(3\cosh 6 + \sinh 6)} - x^2 - \frac{2}{9}.$$

(2) Riešte okrajovú úlohu

$-(1 + x^2)u'' + 2xu' = 1,\ 0 < x < \frac{1}{2},\ u'(0) = 2,\ u(1) = 0.$

$[(1 + x^2)u']' = -1 \Rightarrow (1 + x^2)u' = -x + c_1 \ \Rightarrow u'(x) = \frac{-x + c_1}{1 + x^2}.$

$u'(0) = c_1 = 2 \ \Rightarrow\ u'(x) = -\frac{x}{1+x^2} + \frac{2}{1+x^2},$

$$u(x) = -\frac{1}{2}\int \frac{2x}{1 + x^2}\,dx + \int \frac{2}{1 + x^2}\,dx$$

$$= -\frac{1}{2}\ln(1 + x^2) + 2\arctan x + c_2.$$

$u(1) = -\frac{1}{2}\ln 2 + \frac{\pi}{2} + c_2 = 0 \ \Rightarrow\ c_2 = \frac{1}{2}\ln 2 - \frac{\pi}{2}.$

$$u(x) = -\frac{1}{2}\ln(1+x^2) + 2\arctan x + \frac{1}{2}\ln 2 - \frac{\pi}{2}$$
$$= \ln\sqrt{\frac{2}{1+x^2}} + 2\arctan x - \frac{\pi}{2}.$$

(3) Riešte úlohu na vlastné hodnoty a vlastné funkcie.

a) $u'' + \lambda u = 0,\ 0 < x < 2\pi,\ u'(0) = u'(2\pi) = 0.$

Neumannova okrajová úloha :
$\lambda_0 = 0,\ u_0(x) \equiv 1.$
$\lambda > 0,\ u(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x,$
$u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x,$
$u'(0) = \sqrt{\lambda}c_2 = 0,\ \Rightarrow c_2 = 0,\ c_1 = 1.$
$u(x) = \cos\sqrt{\lambda}x.$
$u'(2\pi) = \sqrt{\lambda}\sin\sqrt{\lambda}2\pi = 0 \Rightarrow \sqrt{\lambda}2\pi = n\pi,\ \sqrt{\lambda} = \frac{n}{2}.$

$$\lambda_n = \frac{n^2}{4},\ u_n(x) = \cos\frac{n}{2}x,\ n \in \mathbb{N}_0.$$

b) $u'' + \lambda u = 0,\ 0 < x < 2\pi,\ u'(0) = u(2\pi) = 0.$

Zmiešaná okrajová úloha :
$\lambda > 0,\ u(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x,$
$u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x,$
$u'(0) = \sqrt{\lambda}c_2 = 0,\ \Rightarrow c_2 = 0,\ c_1 = 1.$
$u(x) = \cos\sqrt{\lambda}x.$
$u(2\pi) = \cos\sqrt{\lambda}2\pi = 0 \Rightarrow \sqrt{\lambda}2\pi = (2n-1)\frac{\pi}{2},\ \sqrt{\lambda} = \frac{2n-1}{4}.$

$$\lambda_n = \frac{(2n-1)^2}{16},\ u_n(x) = \cos\frac{2n-1}{4}x,\ n \in \mathbb{N}.$$