Osovo symetrické kmitanie na kruhovej oblasti

$$u_{tt} = c^2 \Delta u$$

Prepis Δ do polárnych súradníc

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}\right)$$

Osová symetria $u_\varphi = 0$ a riešenie

$u(r,\varphi,t) = u(r,t)$

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r} u_r\right)$$

Podmienka na okraji kruhu

$u(r_0,t) = f(t) = 0$ Homogénna

$u_r(0,t) = 0$ Symetria

Začiatočné podmienky

$u(r,0) = \varphi(r)$ $u_t(r,0) = \psi(r)$ kvôli zjednodušeniu $\psi(r) = 0$

Metóda separácie

$$u(r,t) = R(r)T(t)$$

$$T''R = c^2\left(R'' + \frac{1}{r}R'\right)T$$

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{1}{R}\frac{R'}{r} = -\lambda$$

$$rR'' + R' + \lambda r R = 0$$

$R(r_0) = 0$ $R'(0) = 0$

Rovnica $\quad tx'' + x' + \lambda t x = 0$

$$x(t_0) = 0 \qquad x'(0) = 0$$

je Besselova dif rovnica.

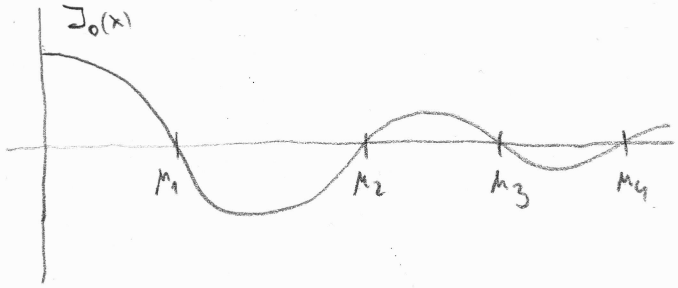
Riešime ju metódou zovšeobecnených mocninových radov $\quad x(t) = \sum_{n=0}^{\infty} c_n x^{n+\nu}$

Dostaneme riešenie

$$x(t) = c_0 - \frac{\lambda c_0}{2^2} t^2 + \frac{\lambda^2 c_0}{2^2 4^2} t^4 - \dots = c_0 \sum_{m=0}^{\infty} \frac{(\sqrt{\lambda} t)^{2m}}{2^{2m} (m!)^2} = c_0 J_0(\sqrt{\lambda} t)$$

Teda riešenia našej úlohy sú

$$R(r) = J_0(\sqrt{\lambda} r)$$



$$\sqrt{\lambda_k} r_0 = \mu_k$$
$$\lambda_k = \left(\frac{\mu_k}{r_0}\right)^2$$

$\mu_1 = 2.4048$

$\mu_2 = 5.5201$

$\mu_3 = 8.6537$

$\mu_4 = 11.7915$

$\vdots$

$$R_k(r) = J_0\left(\mu_k \frac{r}{r_0}\right)$$

$$T'' + \lambda_k c^2 T = 0 \qquad \pm i \sqrt{\lambda_k} e \qquad \text{vlastné hodnoty}$$

$$T_k(t) = A_k \cos \frac{\mu_k c}{r_0} t + B_k \sin \frac{\mu_k c}{r_0} t$$

$$w(r,t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{\mu_k c}{r_0} t + b_k \sin \frac{\mu_k c}{r_0} t \right) J_0\left(\mu_k \frac{r}{r_0}\right)$$

u

$$u(r,0) = \sum_{k=1}^{\infty} a_k \, J_0\left(\mu_k \frac{r}{r_0}\right)$$

$$a_k = \frac{\int_0^{r_0} \varphi(r) \, J_0\left(\mu_k \frac{r}{r_0}\right) dr}{\int_0^{r_0} \left(J_0\left(\mu_k \frac{r}{r_0}\right)\right)^2 dr}$$

$b_k = 0$ lebo $\varphi(r) = 0$