Metóda charakteristických súradníc.

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = f(x,y)$$

Zavedieme nový súradný systém „pozdĺž charakteristík".

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

predpokladajme, že po vyriešení tejto rovnice jej riešenie $y(x)$ spĺňa vzťah

$$h(x, y(x)) = c$$

Nové súradnice $\xi = h(x,y)$

$\tau = y$

Derivácie
$$u_x = w_\xi \xi_x + w_\tau \tau_x = w_\xi h_x$$
$$u_y = w_\xi \xi_y + w_\tau \tau_y = w_\xi h_y + w_\tau$$

Rovnica $a(x,y)\, w_\xi h_x + b(x,y)(w_\xi h_y + w_\tau) + c(x,y)\,w = f(x,y)$

(trochu zmiešané označenie premenných)

Pritom

$h(x, y(x)) = c$ znamená

$$h_x(x,y(x)) + h_y(x,y(x))\cdot\frac{dy}{dx} = 0$$

$$h_x + h_y\frac{b(x,y)}{a(x,y)} \Rightarrow \underline{h_x\, a(x,y) = -h_y\, b(x,y)},$$

Naspäť do rovnice:

$$b(\xi,\tau)\, w_\tau + c(\xi,\tau)\, w = f(\xi,\tau)$$

Toto je nehomogénna LDR 1. rádu v premennej τ (v nej je ξ parameter)

Po jej vyriešení dostaneme riešenie PDR, transformáciou do pôvodných súradníc

Príklad 1 $\quad u_x + y u_y = y e^y$

Charakteristiky $\quad y'(x) = \frac{y}{1} \qquad y(x) = c e^x \qquad y e^{-x} = c$

Nové súradnice $\quad \xi = y e^{-x}$

$\tau = y$

Rovnica

$$-u_\xi y e^{-x} + y u_\xi e^{-x} + y w_\tau = y e^y$$
$$\tau w_\tau = \tau e^\tau$$
$$w_\tau = e^\tau$$
$$w = e^\tau + c(\xi)$$

$$\underline{w(x,y) = e^y + c(y e^{-x})}$$

Príklad 2 $\quad w_x + y u_y = y e^x$

$\vdots$

$y u_\tau = y e^x \qquad \xi = \tau e^{-x} \qquad \frac{\tau}{\xi} = e^x$

$u_\tau = \frac{\tau}{\xi}$

$u = \frac{\tau^2}{2\xi} + c(\xi)$

$$w(x,y) = \frac{y^2}{2y e^{-x}} + c(y e^{-x}) = \underline{\frac{1}{2} y e^x + c(y e^{-x})}$$

Princíp superpozície

Príklad 3 $\quad w_x + y u_y = y(e^x + e^y)$

$$\underline{w(x,y) = e^y + \frac{1}{2} y e^x + c(y e^{-x})}$$

Príklad 4 $\quad w_x + y w_y = y e^y$

$w(0,y) = \sin y$

$u(x,y) = e^y + c(y e^{-x})$

$w(0,y) = e^y + c(y) = \sin y \qquad c(y) = \sin y - e^y$

$c(w) = \sin w - e^w$

$$\underline{u(x,y) = e^y + \sin(y e^{-x}) - e^{y e^{-x}}}$$

Veta (o existencii riešenia)

Rovnica

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = f(x,y)$$

má pre okrajovú podmienku

$$u(x_0(s), y_0(s)) = u_0(s)$$

zadanú na krivke $\gamma = (x_0(s), y_0(s))$ pre $s \in I$

pri splnenej podmienke regularity

$$\frac{\partial x_0}{\partial s} \cdot b(\gamma(s)) - \frac{\partial y_0}{\partial s}\, a(\gamma(s)) \neq 0 \qquad \left[\gamma'(s)\cdot(b,-a) \neq 0\right]$$

jediné riešenie $u(x,y)$ definované na okolí krivky γ

Príklad 5 $\quad x u_x - y u_y + y u = x y^2$

s podmienkou

$$u(1,y) = y-1$$

Charakteristiky

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$-\frac{dy}{y} = \frac{dx}{x} \qquad \text{predpoklad } x,y \neq 0$$

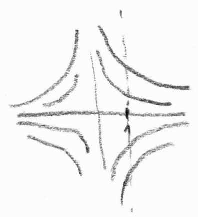
$$-\ln y = \ln x + c$$

$$\ln xy = \ln x + \ln y = -c$$

$$xy = e^{-c} = \tilde{c}$$

Súradnice $\quad \xi = xy$

$\tau = y$



$$x \cancel{w_\xi y} - y \cancel{w_\xi x} - y w_\tau + y w = x y^2$$

$$-w_\tau + w = xy = \xi$$

Homogénna $\quad w_H(\xi,\tau) = C(\xi)\cdot e^\tau$

$$w_p(\xi,\tau) = \left(\int_0^\tau \frac{\xi}{e^\tau}\, d\tau\right) e^\tau = -\xi(e^{-\tau}-1)\cdot e^\tau = \xi(e^\tau - 1)$$

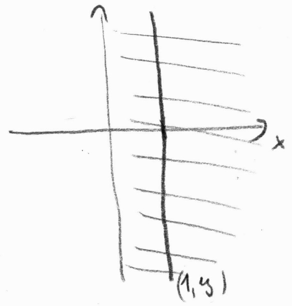
$$w(x,y) = C(xy)e^y + xy(e^y - 1) = C_1(xy)e^y - xy$$

$w(x,y) = c_1(xy)e^{y} - xy$

$w(1,y) = c_1(y)e^{y} - y = y-1$ $\qquad c_1(y) = (2y-1)e^{-y}$

$\underline{w(x,y) = (2xy-1)e^{-xy}\cdot e^{y} - xy} = \underline{(2xy-1)\,e^{y(1-x)} - xy}$

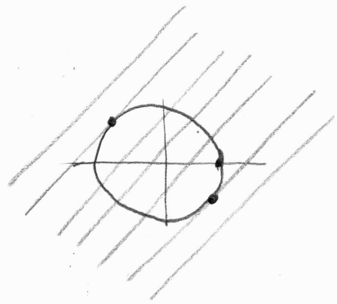
Riešenie sa dá rozšíriť aj na $x=0$



(aj na $y=0$

Príklad 6. $\qquad w_x + w_y = 0$

s podmienkou $\qquad w(\cos s, \sin s) = s$



Podmienka regularity nie je splnená pre $s = \frac{3}{4}\pi$ a $s = -\frac{\pi}{4}$

$(-\sin s, \cos s)\cdot(1,-1) = -\sin s - \cos s \neq 0 \qquad \cos s \neq -\sin s$

Riešenie rovnice $\qquad u(x,y) = f(x-y)$.

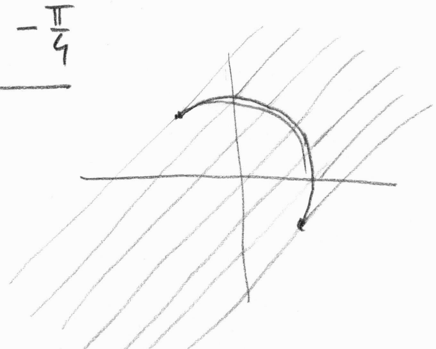
$w(\cos s, \sin s) = f(\cos s - \sin s) = s$ $\qquad$ Uvažujeme $s \in (-\frac{\pi}{4}, \frac{3\pi}{4})$

$= f\left(\sqrt{2}\cos\left(s+\frac{\pi}{4}\right)\right) = s$

$= f(w) = s$ $\qquad w = \sqrt{2}\cos(s+\frac{\pi}{4})$

$\arccos\frac{w}{\sqrt{2}} - \frac{\pi}{4} = s$

$\underline{w(x,y) = \arccos\frac{x-y}{\sqrt{2}} - \frac{\pi}{4}}$



Lineárna PDR 2. rádu
(s konštantnými koeficientami)

Vlnová rovnica

$u_{tt} = c^2 u_{xx}$ uvažujeme ju pre $x \in \mathbb{R}$ a $t \in \mathbb{R}^+$
$c > 0$ konštanta

Prevod na sústavu 2 rovníc 1. rádu

$$0 = u_{tt} - c^2 u_{xx} = \frac{\partial^2}{(\partial t)^2} u - c^2 \frac{\partial^2}{(\partial x)^2} u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

„Operátory"

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = v$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0$$

Sústava
$$u_t + c u_x = v$$
$$v_t - c v_x = 0$$

Riešme druhú $v_t - c v_x = 0$ Charakteristiky $-ct - x$ resp. $x + ct = \text{konšt.}$

Všeobecné riešenie $v(x,t) = \tilde{f}(x+ct)$

Riešme prvú $u_t + c u_x = \tilde{f}(x+ct)$

Homogénna $u_t + c u_x = 0$ má riešenie $u_H(x,t) = g(x-ct)$

a partikulárne riešenie
$$u_p(x,t) = \tilde{F}(x+ct)$$

Sk: $(u_p)_t + c(u_p)_x = c\tilde{F}'(x+ct) + c\tilde{F}'(x+ct) = \tilde{f}(x+ct)$

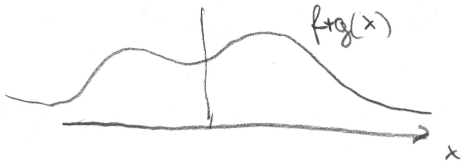
Stačí teda zobrať $\tilde{F}(w) = \int \frac{1}{2c}\tilde{f}(w)\,dw$

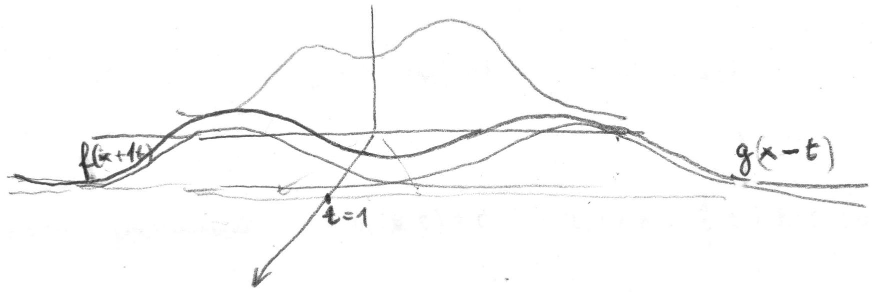
Teraz je $w(x,t) = g(x-ct) + \tilde{F}(x+ct)$

Zvyčajne značíme

$$w(x,t) = f(x+ct) + g(x-ct)$$

Predstava





ALE f+g sa môže rozdeliť na súčet nekonečne veľa spôsobmi

Metóda charakteristických súradníc

$$w_{tt} = c^2 w_{xx}$$

Zvoľme novú súradnú sústavu v rovine v podobe

$$\xi = x+ct$$
$$\tau = x-ct$$

Derivácie

$$w_x = w_\xi \cdot \xi_x + w_\tau \tau_x = w_\xi + w_\tau$$
$$w_t = w_\xi \xi_t + w_\tau \tau_t = c\cdot w_\xi - c\,w_\tau = c(w_\xi - w_\tau)$$

Druhé derivácie

$$w_{xx} = w_{\xi\xi}\cdot 1 + w_{\xi\tau}\cdot 1 + w_{\tau\xi}\cdot 1 + w_{\tau\tau}\cdot 1 = w_{\xi\xi} + 2w_{\xi\tau} + w_{\tau\tau}$$
$$w_{tt} = c\,(w_{\xi\xi}c + w_{\xi\tau}(-c) - w_{\tau\xi}c - w_{\tau\tau}(-c)) = c^2(w_{\xi\xi} - 2w_{\xi\tau} + w_{\tau\tau})$$

Rovnica: $c^2(w_{\xi\xi} - 2w_{\xi\tau} + w_{\tau\tau}) = c^2(w_{\xi\xi} + 2w_{\xi\tau} + w_{\tau\tau})$

$$0 = 4c^2 w_{\xi\tau}$$