Stabilita pevných bodov.

Systém $\vec{x}' = F(\vec{x})$

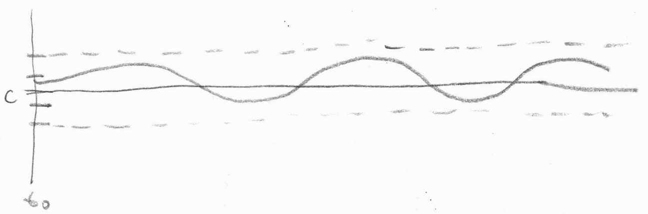
Pevný bod c

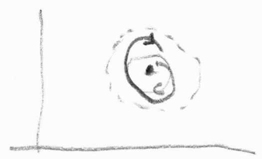
Pevný bod c nazývame stabilný, ak $\forall \varepsilon > 0$ existuje také okolie $O_\delta(c)$ že $\forall x_1 \in O_\delta(c)$ je riešenie úlohy

$$\vec{x}' = F(\vec{x})$$
$$x(t_0) = x_1$$

v $O_\varepsilon(c)$.

$$\left(|x_1 - x_0| < \delta \Rightarrow |x(t) - c_0| < \varepsilon \right)$$





Pevný bod c nazývame asymptoticky stabilný ak

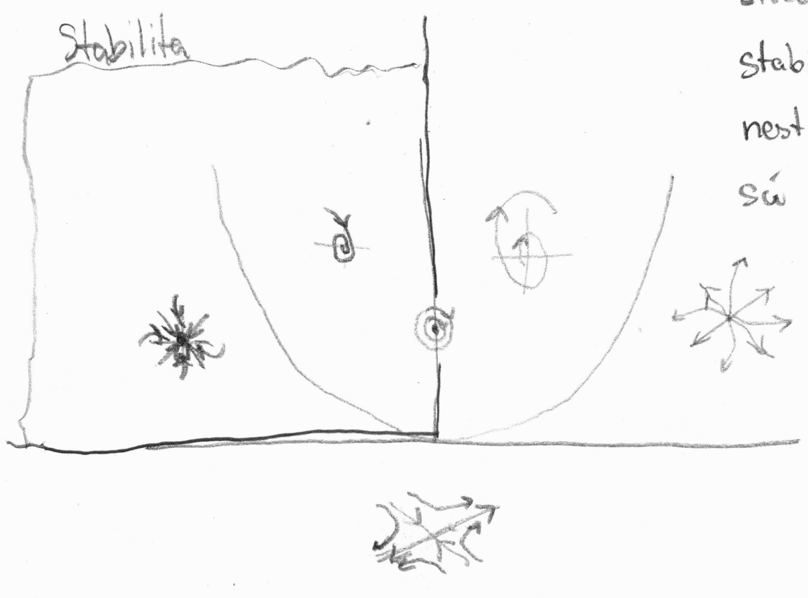
je stabilný

pre riešenia $x(t)$ s $x(t_0) \in O_\delta(c)$ je $\lim_{t \to \infty} x(t) = c$

V lineárnych systémoch je asymptoticky stabilný pevný bod stoka a stabilné ohnisko.

stabilný je stred

nest. ohnisko, žriedlo a sedlo sú nestabilné pevné body



Ak c je pevný bod systému $\vec{x}' = F(\vec{x})$ a
$\vec{u}' = DF(c)\vec{u}$ linearizácia v bode c,
tak c je asymptoticky stabilný ak $\mathrm{Re}\,\lambda_i < 0$ $\forall \lambda_i$ vlastné číslo linearizácie.
Ak $\exists \lambda_i$ také, že $\mathrm{Re}\,\lambda_i > 0$ tak c je nestabilný pevný bod.

Príklad

$$x' = -1.5\,xy + 1$$
$$y' = 1.5\,xy - y$$

Pevné body
$$-1.5\,xy + 1 = 0$$
$$1.5\,xy - y = 0 \qquad y(1.5x - 1) = 0$$

$y = 0$ nerieši 1. rovnicu $\qquad 1.5x - 1 = 0 \qquad x = \frac{2}{3}$

$$1 - y = 0 \qquad y = 1$$

Pevný bod $\left[\frac{2}{3}, 1\right]$

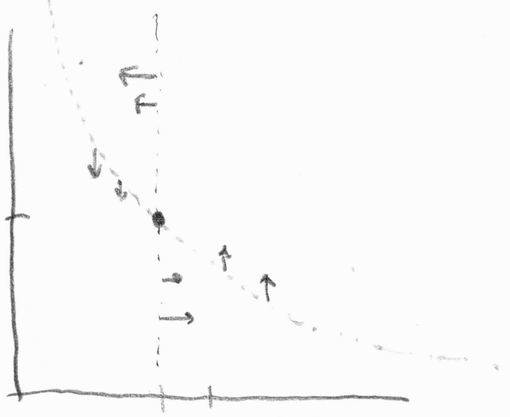
Linearizácia
$$\begin{pmatrix} -1.5y & -1.5x \\ 1.5y & 1.5x - 1 \end{pmatrix}$$

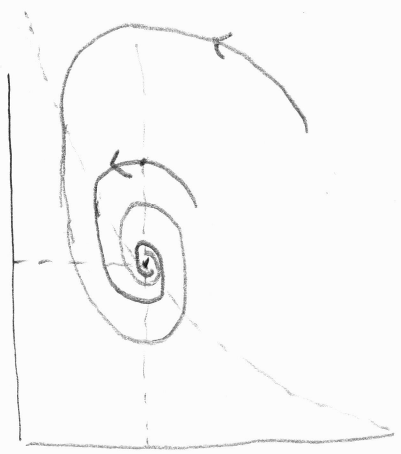
v pevnom bode
$$\begin{pmatrix} -1.5 & -1 \\ 1.5 & 0 \end{pmatrix} \qquad \begin{vmatrix} -1.5-\lambda & -1 \\ 1.5 & -\lambda \end{vmatrix} = \lambda^2 + 1.5\lambda + 1.5 = 0$$

$$\lambda_{1,2} = \frac{-1.5 \pm \sqrt{\frac{9}{4} - 6}}{2} = -\frac{3}{4} \pm i\frac{\sqrt{15}}{4}$$

stabilné ohnisko

Izokliny





Konzervatívne systémy a 1. integrál

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad (S)$$

Spolu so systémom uvažujme (skalárnu) funkciu $F(x,y)$, $F: \mathbb{R}^2 \to \mathbb{R}$
F - nekonštantná (na oblastiach v $\mathbb{R}^2$)

Def. Nekonštantnú funkciu F nazývame 1. integrál systému (S), ak pre každé riešenie $(x(t), y(t))$ systému (S) existuje konštanta c taká, že $F(x(t), y(t)) = c$

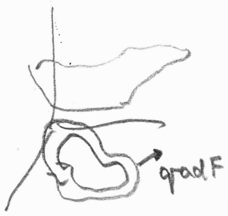
Predstava: F chápeme ako hodnotu vyčíslovanú pozdĺž riešenia $x(t), y(t)$, ktorá sa s časom nemení (je konzervovaná)

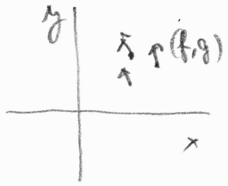
Matematicky $\frac{\partial F}{\partial t}(x(t), y(t)) = 0$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)\cdot\left(x'(t), y'(t)\right) = \operatorname{grad} F(x(t),y(t)) \cdot (x'(t), y'(t)) = 0$$

Teda $\quad \operatorname{grad} F \cdot (f,g) = 0$

F





Vektorové pole (f,g) je kolmé na $\operatorname{grad} F$ teda riešenie $(x(t), y(t))$ leží na vrstevnici funkcie F.

Príklad.

$$x' = y$$
$$y' = -x$$

a funkcia

$$F(x,y) = x^2 + y^2$$

Overenie cez riešenia $\varphi_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ $\varphi_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Všeobecné

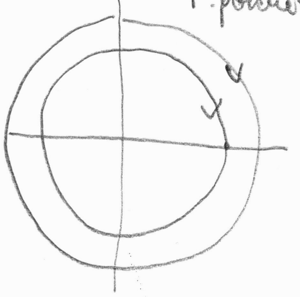
$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

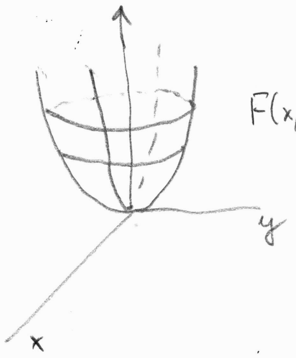
$$F(x(t), y(t)) = (c_1 \cos t + c_2 \sin t)^2 + (c_1(-\sin t) + c_2 \cos t)^2 = c_1^2 + 2c_1c_2 \cos t \sin t - 2c_1c_2 \cos t \sin t + c$$
$$= c_1^2 + c_2^2 = c \text{ - (amplitúda)}$$

Overenie cez gradient $\operatorname{grad} F = (2x, 2y)$

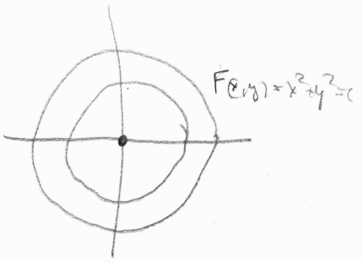
$$\operatorname{grad} F \cdot (f,g) = (2x,2y)\cdot(y,-x) = 2xy - 2xy = 0$$

F. podmnet systému





vrstevnice



Nejednoznačnosť

$F_1(x,y) = e^{x^2+y^2}$ je tiež 1 integrál

$$\operatorname{grad} F_1 = \left(e^{x^2+y^2}\cdot 2x,\ e^{x^2+y^2}\cdot 2y\right)$$
$$\operatorname{grad} F_1 \cdot (f,g) = \left(e^{x^2+y^2} 2x,\ e^{x^2+y^2}\cdot 2y\right)(y,-x) = 0$$

Každá funkcia $\tilde{F}(x,y) = \varphi(x^2+y^2)$ je 1. integrál

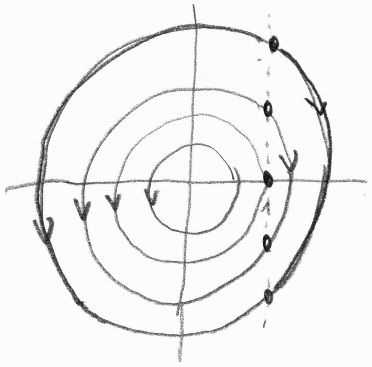
$$\operatorname{grad} \tilde{F} = \left(\varphi'(x^2+y^2)\cdot 2x,\ \varphi'(x^2+y^2)\cdot 2y\right)$$
$$\operatorname{grad} \tilde{F} \cdot (f,g) = 0$$

Naopak $F(x,y)=x^2+y^2$ je 1. integrál pre viaceré systémy dif. rovníc

Okrem $\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$ aj $\begin{aligned} x' &= y(x-1) \\ y' &= -x(x-1) \end{aligned}$

Zrejme $\operatorname{grad} F = (2x, 2y)$

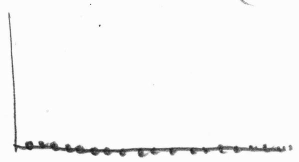
$$(2x,2y)\,\big(y(x-1), -x(x-1)\big) = 0$$



Otázka: Ako nájsť aspoň jeden 1. integrál systému $\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$?

<u>Príklad</u> $\begin{aligned} x' &= -2xy \\ y' &= 2xy - y \end{aligned}$

Pevné body $\begin{aligned} -2xy &= 0 \\ (2x-1)\,y &= 0 \end{aligned}$ $y=0$ x - ľubovoľné $[x_0, 0]$



$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2xy-y}{-2xy} = -1 + \frac{1}{2x}$$

$$\frac{dy}{dx} = y'(x) = -1 + \frac{1}{2x}$$ Separácia

$$\int dy = \int -1 + \frac{1}{2x}\, dx$$

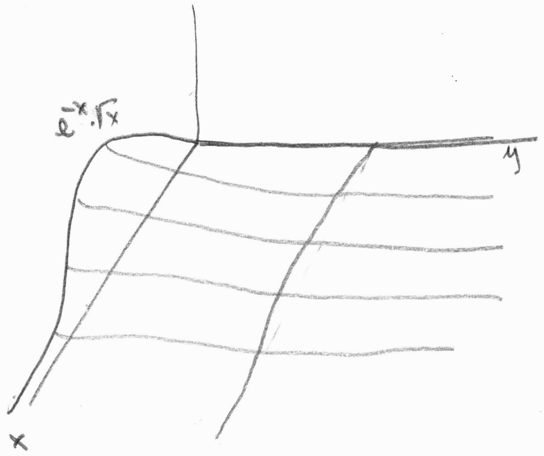
$$y(x) = -x + \frac{1}{2}\ln x + c$$

$$y + x - \frac{1}{2}\ln x = c$$ Ľavá strana je konštantná

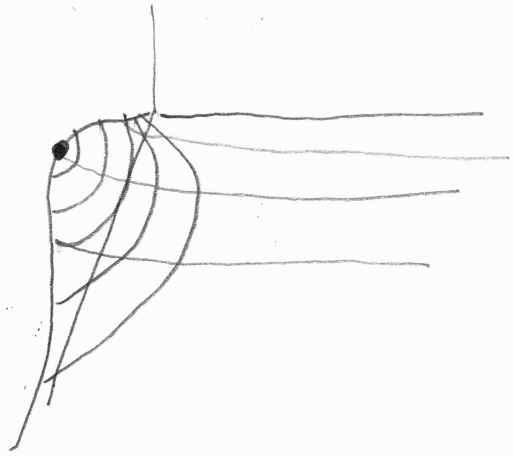
$$F(x,y) = y + x - \frac{1}{2}\ln x$$ je 1. integrál

Kontrola $\quad \operatorname{grad} F \cdot f = \left(1 - \frac{1}{2x}, 1\right)\left(-2xy, 2xy - y\right) = -2xy + y + 2xy - y = 0$

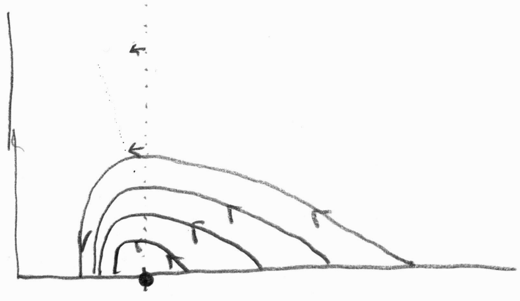
Aj $\quad e^{-F(x,y)} = e^{-y-x+\frac{1}{2}\ln x} = e^{-y}\cdot e^{-x}\cdot\sqrt{x}$ $\quad$ je 1. integrál.



Vrstevnice



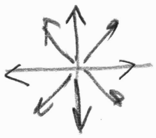
$x' = -2xy$
$y' = 2xy - y$



Globálny fázový portrét

Nie každý systém je konzervatívny

$x' = x$
$y' = y$



... pevný bod typu žriedlo

Funkcia konštantná pozdĺž riešení by bola konštantná v R^2.

Ale $F(x,y)=c$ nie je 1. integrál a ani nedáva žiadnu informáciu o riešeniach.

Príklad: Model Dravec - Korisť

$x(t)$ – populácia koristi v čase t

$y(t)$ – –||– dravca –||–

$$x' = ax - bxy$$
$$y' = -cy + dxy \qquad a,b,c,d>0$$

Hľadanie 1. integrálu

$$\frac{dy}{dx} = \frac{(-c+dx)\,y}{(a-by)\,x}$$

$$\int \frac{a-by}{y}\,dy = \int \frac{-c+dx}{x}\,dx$$

$$a\int \frac{1}{y}\,dy - by = -c\int \frac{1}{x}\,dx + dx$$

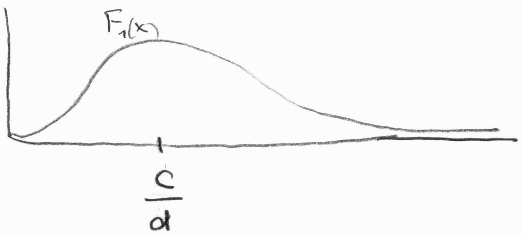
$$a\ln y - by = -c\ln x + dx + \text{konšt}$$

$$a\ln y + c\ln x - by - dx = \text{konšt}$$

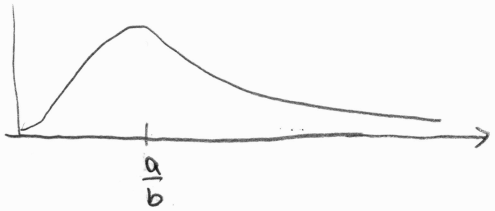
$$e^{\ln y^a} \cdot e^{\ln x^c} \cdot e^{-by} \cdot e^{-dx} = e^{\text{konšt}} = k$$

$$F(x,y) = y^a x^c \cdot e^{-by} \cdot e^{-dx} = F_1(x) \cdot F_2(y)$$

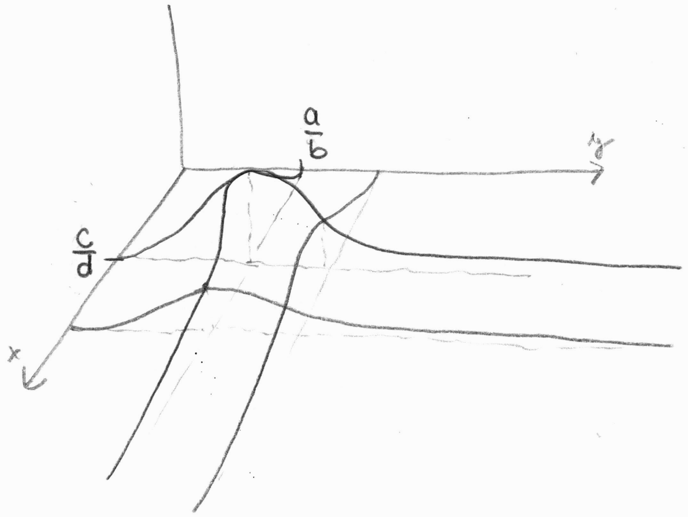
Ako vyzerá $F_1(x) = x^c e^{-dx}$?



$F_2(y)$



$F(x,y) = F_1(x) \cdot F_2(y)$

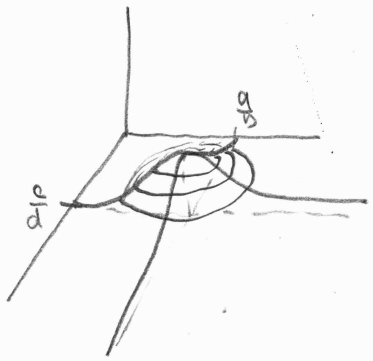


Pevné body:

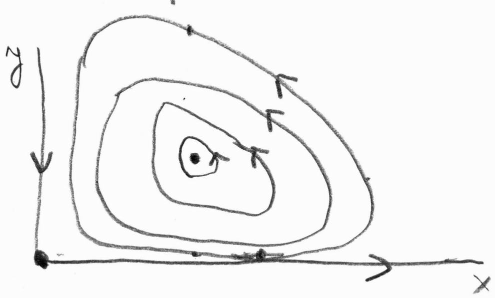
$$ax - bxy = 0 \qquad x = 0 \quad y = 0$$
$$-cy + dxy = 0$$

$$a - by = 0$$
$$-c + dx = 0 \qquad \left[\frac{c}{d}, \frac{a}{b}\right] \qquad [0,0]$$

Vrstevnice 1. integrálu

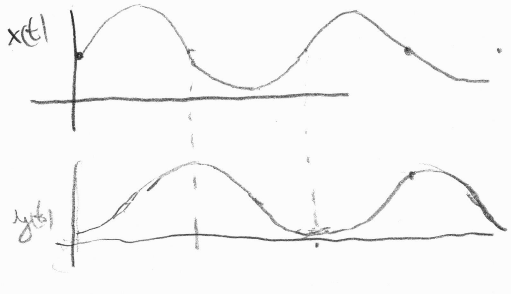


Fázový portrét



$[0,0]$ pevný bod typu sedlo

$\left[\frac{c}{d}, \frac{a}{b}\right]$ pevný bod typu stred



Systém

Dravec - korisť

$$x' = x - xy$$
$$y' = -y + xy$$