Cvičenie z M3 - štvrtý týždeň zima 2020/21
Fourierova transformácia

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1 Dodatok ku M3cvic4

Niektoré úlohy nezaradené do M3Cvic4.pdf, alebo riešené iným spôsobom

12. Pomocou Fourierovho obrazu funkcie $f(t)$ určte Fourierov obraz funkcie $g(t) = f(3t-2)$. $\left[\hat{g}(p) = e^{-\frac{2}{3}ip}\frac{1}{3}\hat{f}(\frac{p}{3})\right]$

    $\mathcal{F}\{f(3t-2)\} = \int_{-\infty}^{\infty} f(3t-2)e^{-ipt}dt = |u = 3t-2| =$

    $= \int_{-\infty}^{\infty} f(u)e^{-\frac{ip(u+2)}{3}}\frac{1}{3}du = \frac{1}{3}e^{-ip\frac{2}{3}}\int_{-\infty}^{\infty} f(u)e^{-i\frac{p}{3}u}du = \frac{1}{3}e^{-i\frac{2}{3}p}\hat{f}(\frac{p}{3})$,
    alebo skrátene

    $\mathcal{F}\{f(3\left(t-\frac{2}{3}\right))\} = e^{-i\frac{2}{3}p}\mathcal{F}\{f(3\,(u))\} = e^{-i\frac{2}{3}p}\frac{1}{3}\hat{f}(\frac{p}{3})$.

13. Pomocou Fourierovho obrazu funkcie $f(t)$ určte Fourierov obraz funkcie $g(t) = tf(2t+1)$. $\left[\hat{g}(p) = \frac{i}{2}e^{i\frac{p}{2}}\left(\frac{i}{2}\hat{f}(\frac{p}{2}) + \frac{1}{2}f\,'(\frac{p}{2})\right)\right]$

    $\mathcal{F}\{tf(2t+1)\} = \int_{-\infty}^{\infty} tf(2t+1)e^{-ipt}dt = |u = 2t+1| = \int_{-\infty}^{\infty} \frac{u-1}{2}f(u)e^{-ip\frac{u-1}{2}}\frac{dt}{2} =$

    $= \frac{1}{4}e^{i\frac{p}{2}}\int_{-\infty}^{\infty} uf(u)e^{-ip\frac{u}{2}}dt - \frac{1}{4}e^{i\frac{p}{2}}\int_{-\infty}^{\infty} f(u)e^{-ip\frac{u}{2}}dt = \frac{1}{4}e^{i\frac{p}{2}}\left(\int_{-\infty}^{\infty} uf(u)e^{-i\frac{p}{2}u}dt - \int_{-\infty}^{\infty} f(u)e^{-i\frac{p}{2}u}d\right.$

    $= \frac{i}{4}e^{i\frac{p}{2}}\left(\frac{d}{dp}\hat{f}\left(\frac{p}{2}\right) + i\hat{f}\left(\frac{p}{2}\right)\right)$

14. Pomocou Fourierovho obrazu funkcie $f(t)$ určte Fourierov obraz funkcie $g(t) = e^{-it}f\,'(2t-1)$. $\left[\hat{g}(p) = \frac{1}{2}e^{-\frac{1}{2}i(p+1)}i\frac{p+1}{2}\hat{f}(\frac{p+1}{2})\right]$

$\mathcal{F}\{e^{-it}f\,'(2t{-}1)\} = \int_{-\infty}^{\infty} e^{-it}f\,'(2t{-}1)e^{-ipt}dt = \int_{-\infty}^{\infty}\left(f\,'(2t-1)\right)e^{-i(p+1)t}dt =$
$|u = 2t-1| =$

$= \int_{-\infty}^{\infty} f\,'(u)e^{-i(p+1)\frac{u+1}{2}}\frac{du}{2} = \frac{1}{2}\int_{-\infty}^{\infty} f\,'(u)e^{-i\left(\frac{(p+1)}{2}u+\frac{(p+1)}{2}\right)}du =$

$= \frac{1}{2}e^{-i\frac{(p+1)}{2}}\int_{-\infty}^{\infty} f\,'(u)e^{-i\frac{(p+1)}{2}u}du = \frac{1}{2}e^{-\frac{1}{2}i(p+1)}i\frac{p+1}{2}\int_{-\infty}^{\infty} f(u)e^{-i\frac{(p+1)}{2}u}dt =$
$\frac{1}{2}e^{-\frac{1}{2}i(p+1)}i\frac{p+1}{2}\overset{\curlywedge}{f}(\frac{p+1}{2}).$

16. Pomocou vety o rezíduách vypočítajte $\displaystyle\int\limits_{-\infty}^{\infty}\frac{1}{(t^2+1)(t^2+2)}e^{-ipt}dt.$

Pomocou tohto výsledku určte Fourierov obraz nasledujúcich funkcií:

1) $f(t) = \frac{1}{(4t^2+1)(4t^2+2)}$,

2) $g(t) = \frac{d}{dt}\frac{1}{(t^2+1)(t^2+2)}$.

$$\left[\begin{array}{l}\pi\left(e^{-|p|}-\frac{1}{\sqrt{2}}e^{-\sqrt{2}|p|}\right),\\ 1)\ \frac{\pi}{2}e^{-\frac{|p|}{2}}-\frac{1}{\sqrt{2}}e^{-\sqrt{2}\frac{|p|}{2}},\\ 2)\ ip\pi\left(e^{-|p|}-\frac{1}{\sqrt{2}}e^{-\sqrt{2}|p|}\right)\end{array}\right]$$

$\frac{1}{(t^2+1)(t^2+2)}$

$$p\neq 0: R(z) = \frac{1}{\left(\left(-\frac{z}{p}\right)^2+1\right)\left(\left(\left(-\frac{z}{p}\right)^2+2\right)\right)} = \frac{p^4}{(z^2+p^2)(z^2+2p^2)} = \frac{p^4}{(z-i|p|)(z+i|p|)\left(z-i|p|\sqrt{2}\right)\left(z+i|p|\sqrt{2}\right)}$$

$$F(p) = \overset{\curlywedge}{f}(p) = \int\limits_{-\infty}^{\infty}\frac{1}{(t^2+1)(t^2+2)}e^{-ipt}dt = \frac{2\pi i}{|p|}\sum res_{\left\{z\,:\,Q\left(-\frac{z}{p}\right)=0,\operatorname{Im}z>0\right\}}\left(R(z)\,e^{iz}\right) =$$

$$= \frac{2\pi i}{|p|}\sum res_{\left\{z\,:\,Q\left(-\frac{z}{p}\right)=0,\operatorname{Im}z>0\right\}}\left(\frac{p^4}{(z-i|p|)(z+i|p|)\left(z-i|p|\sqrt{2}\right)\left(z+i|p|\sqrt{2}\right)}e^{iz}\right) =$$

$$= \frac{2\pi i}{|p|}\left[res_{z=i|p|}\left(z-i\,|p|\right)\frac{p^4}{(z-i|p|)(z+i|p|)\left(z-i|p|\sqrt{2}\right)\left(z+i|p|\sqrt{2}\right)}e^{iz} + res_{z=i|p|\sqrt{2}}\left(z-i\,|p|\,\sqrt{2}\right)\frac{}{(z-i|p|)(z+i}\right.$$

$$= \frac{2\pi i}{|p|}\lim_{z\to i|p|}\frac{p^4}{(z+i|p|)\left(z-i|p|\sqrt{2}\right)\left(z+i|p|\sqrt{2}\right)}e^{iz}+\frac{2\pi i}{|p|}\lim_{z\to i|p|\sqrt{2}}\frac{p^4}{(z+i|p|)(z+i|p|)\left(z+i|p|\sqrt{2}\right)}e^{iz} =$$

$$= \frac{2\pi i}{|p|}\frac{p^4}{(2i|p|)(i|p|)^2\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}e^{ii|p|} + \frac{2\pi i}{|p|}\frac{p^4}{\left(2i|p|\sqrt{2}\right)(i|p|)^2}e^{ii|p|\sqrt{2}} = \frac{\pi}{1}e^{-|p|} -$$
$$\frac{\pi}{\sqrt{2}}e^{-|p|\sqrt{2}} = \pi\left(e^{-|p|}-\frac{1}{\sqrt{2}}e^{-|p|\sqrt{2}}\right)$$

$$p = 0: \int\limits_{-\infty}^{\infty}\frac{1}{(t^2+1)(t^2+2)}dt = 2\pi i\sum res_{\{z\,:\,Q(z)=0,\operatorname{Im}z>0\}}\left(\frac{1}{(z^2+1)(z^2+2)}\right) =$$

$$= 2\pi i\left[res_{z=i}\left(\frac{1}{(z^2+1)(z^2+2)}\right) + res_{z=i\sqrt{2}}\left(\frac{1}{(z^2+1)(z^2+2)}\right)\right] =$$

$$= 2\pi i \left[\lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)\left(z-i\sqrt{2}\right)\left(z+i\sqrt{2}\right)} + \lim_{z\to i\sqrt{2}}\left(z-\sqrt{2}i\right)\frac{1}{(z-i)(z+i)\left(z-i\sqrt{2}\right)\left(z+i\sqrt{2}\right)}\right] =$$

$$= 2\pi i \left[\lim_{z\to i}\frac{1}{(z+i)\left(z-i\sqrt{2}\right)\left(z+i\sqrt{2}\right)} + \lim_{z\to i\sqrt{2}}\frac{1}{(z-i)(z+i)\left(z+i\sqrt{2}\right)}\right] =$$

$$= 2\pi i \left[\lim_{z\to i}\frac{1}{(2i)(i)^2\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)} + \lim_{z\to i\sqrt{2}}\frac{1}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)(i)^2\left(2i\sqrt{2}\right)}\right] =$$

$$2\pi i \left[\frac{-i}{2} + \frac{i}{\left(2\sqrt{2}\right)}\right] =$$

$$= -\frac{1}{\sqrt{2}}\pi\left(1-\sqrt{2}\right) = \pi\left(1-\frac{1}{\sqrt{2}}\right).$$

Bolo to možné vypočítať aj cez spojitosť: $\lim_{p\to 0}\pi\left(e^{-|p|} - \frac{1}{\sqrt{2}}e^{-|p|\sqrt{2}}\right) =$
$\pi\left(1-\frac{1}{\sqrt{2}}\right).$

Pomocou tohto výsledku určte Fourierov obraz nasledujúcich funkcií:

1) $f(t) = \frac{1}{(4t^2+1)(4t^2+2)}$,

$$\mathcal{F}\left\{\frac{1}{(t^2+1)(t^2+2)}\right\} = \mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty}\frac{1}{(t^2+1)(t^2+2)}e^{-ipt}dt = \pi\left(e^{-|p|} - \frac{1}{\sqrt{2}}e^{-|p|\sqrt{2}}\right),$$

potom

$$\mathcal{F}\left\{\frac{1}{(4t^2+1)(4t^2+2)}\right\} = \mathcal{F}\left\{f(2t)\right\} = \frac{1}{2}\pi\left(e^{-\left|\frac{p}{2}\right|} - \frac{1}{\sqrt{2}}e^{-\left|\frac{p}{2}\right|\sqrt{2}}\right)$$

2) $\mathcal{F}\left\{\frac{d}{dt}\frac{1}{(t^2+1)(t^2+2)}\right\} = ip\pi\left(e^{-|p|} - \frac{1}{\sqrt{2}}e^{-|p|\sqrt{2}}\right)$