Skúška z M2E

Meno a priezvisko: **16. júna 2022**

	1.	2.	3.	4.	$\sum$	$\sum_{T+P}$	$\sum_S$	$\sum_{T+P+S}$	Známka
Príklady									

1. Vypočítajte integrál $\int \frac{e^x+10}{(e^{2x}-2e^x+5)}\,dx$. (**20b**)

2a. a. Vypočítajte smer, v ktorom je smerová derivácia funkcie $f(x,y) = \ln\frac{x+y}{x-y}$, v bode $\boldsymbol{a} = (3,0)$ maximálna a jej hodnotu. (**5b**),

b. Ukážte, že plochy $x^2 - xy - 8x + z + 5 = 0$ a $4 + x + 2y = \ln z$ sa dotýkajú v bode $T = (2,-3,1)$.(**5b**)

2b. Vyšetrite stacionárne body a vypočítajte lokálne extrémy funkcie $f(x,y) = xy(2-x-y)$. (**10b**)

3. Vypočítajte integrál $\iint_A \left(x^2+y\right) dxdy$, kde A je množina ohraničená krivkami $y = \frac{1}{2}x$, $xy = 2$, $x = 1$. Načrtnite obrázok množiny A. (**20b**)

4. Použitím cylindrických súradníc vypočítajte objem množiny $A = \left\{(x,y,z)\,;\, x^2+y^2 \le 1,\, 0 \le z \le 2-y^2\right\}$. Načrtnite obrázok množiny A. (**20b**)

Riešenie

1. Vypočítajte integrál $\int \frac{e^x+10}{(e^{2x}-2e^x+5)}\, dx$. (**20b**)

$\int \frac{e^x+10}{(e^{2x}-2e^x+5)}\, dx = \left| \begin{array}{ll} t = e^x & x = \ln t \\ dx = \frac{1}{t} dt & \end{array} \right| = \int \frac{t+10}{t(t^2-2t+5)} dt = *$

$\frac{t+10}{t(t^2-2t+5)} = \frac{At+B}{t^2-2t+5} + \frac{C}{t} = \frac{At^2+Bt+Ct^2-2Ct+5C}{t(t^2-2t+5)} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & 10 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow$

$\Rightarrow \frac{t+10}{t(t^2-2t+5)} = \frac{2}{t} + \frac{5-2t}{t^2-2t+5} = \frac{2}{t} - \frac{2t-5}{t^2-2t+5}$

$* = \int \frac{2}{t}\, dt - \int \frac{2t-5}{t^2-2t+5} dt = 2\ln|t| - \int \frac{2t-2-3}{t^2-2t+5} dt = 2\ln|t| - \int \frac{2t-2}{t^2-2t+5} dt +$
$3 \int \frac{1}{t^2-2t+5} dt =$

$= 2\ln|t| - \ln\left|t^2-2t+5\right| + 3\int \frac{1}{(t-1)^2+4} dt =$

$= 2\ln|t| - \ln\left|t^2-2t+5\right| + \frac{3}{4}\int \frac{1}{\left(\frac{t-1}{2}\right)^2+1} dt =$

$= 2\ln|t| - \ln\left|t^2-2t+5\right| + \frac{3}{4} 2\operatorname{arctg}\left(\frac{t-1}{2}\right) =$

$= 2x - \ln\left|e^{2x}-2e^x+5\right| + \frac{3}{2}\operatorname{arctg}\left(\frac{e^x-1}{2}\right) + C$

2a. a. Vypočítajte smer, v ktorom je smerová derivácia funkcie $f(x,y) = \ln\frac{x+y}{x-y}$, v bode $\boldsymbol{a} = (3,0)$ maximálna a jej hodnotu. (**5b**),

b. Ukážte, že plochy $x^2 - xy - 8x + z + 5 = 0$ a $4 + x + 2y = \ln z$ sa dotýkajú v bode $T = (2,-3,1)$.(**5b**)

a. $\frac{\partial f}{\partial x} = \frac{1}{\frac{x+y}{x-y}} \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x^2-y^2)}$, $\frac{\partial f}{\partial y} = \frac{1}{\frac{x+y}{x-y}} \frac{(x-y)+(x+y)}{(x-y)^2} = \frac{2x}{(x^2-y^2)}$, $\frac{\partial f}{\partial x}(3,0) = 0$, $\frac{\partial f}{\partial y}(3,0) = \frac{2}{3}$

$\boldsymbol{e} = (e_1,e_2) \wedge \|\boldsymbol{e}\| = 1$, $\frac{df(\boldsymbol{a})}{d\boldsymbol{e}} = grad\,(\boldsymbol{a}) \circ \boldsymbol{e} = \left(\frac{\partial f(3,0)}{\partial x}, \frac{\partial f(3,0)}{\partial y}\right) \circ \boldsymbol{e} = \left(0, \frac{2}{3}\right) \circ (e_1,e_2) = \frac{2}{3}e_2 \Rightarrow \frac{df(\boldsymbol{a})}{d\boldsymbol{e}}$ bude maximálne pre maximálne e_2

t.j. $e_2 = 1$, tak hľadaný smer je $\boldsymbol{e} = (0,1)$, alebo $\boldsymbol{e} = \frac{grad(\boldsymbol{a})}{\|grad(\boldsymbol{a})\|} = \frac{1}{\|(0,\frac{2}{3})\|}\left(0,\frac{2}{3}\right) = (0,1)$

b. $x^2 - xy - 8x + z + 5 = 0$ a $4 + x + 2y = \ln z$ sa dotýkajú v bode $T = (2,-3,1)$. Ukážeme, že pre $f_1 : z = -5 - x^2 + xy + 8x$ a $f_2 : z = e^{4+x+2y}$ je bod $T = (2,-3,1)$ spoločný dotykový bod a že dotykové roviny v tomto sú rovnaké:

Dotykový bod $(2,-3,1) : z = -5 - 4 - 6 + 16 = 1, z = e^{4+2-6} = 1$

$\frac{\partial f_1}{\partial x} = -2x + y + 8$, $\frac{\partial f_1}{\partial y} = x$, $\frac{\partial f_2}{\partial x} = e^{4+x+2y}$, $\frac{\partial f_2}{\partial y} = 2e^{4+x+2y}$, $\frac{\partial f_1(2,-3)}{\partial x} = 1$, $\frac{\partial f_1(2,-3)}{\partial y} = 2$, $\frac{\partial f_2(2,-3)}{\partial x} = 1$, $\frac{\partial f_2(2,-3)}{\partial y} = 2$, tak

$\tau_1 : z = 1 + 1\,(x-2) + 2\,(y-3)$

$\tau_2 : z = 1 + 1\,(x-2) + 2\,(y-3)$

2b. Vyšetrite stacionárne body a vypočítajte lokálne extrémy funkcie $f(x,y) = xy\,(2-x-y)$. **(10b)**

1° Stacionárne body: $grad f(x,y) = \mathbf{0}$, $\frac{\partial f}{\partial x} = 2y - 2xy - y^2 = y\,(2-2x-y)$, $\frac{\partial f}{\partial y} = 2x - 2xy - x^2 = x\,(2-2y-x)$,

$$\begin{pmatrix} y\,(2-2x-y)=0 \\ x\,(2-2y-x)=0 \end{pmatrix} \Rightarrow \begin{pmatrix} (y=0)\vee(2-2x-y)=0 \\ (x=0)\vee(2-2y-x)=0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x=0 \\ y=0 \end{pmatrix}, \begin{pmatrix} x=0 \\ y=2 \end{pmatrix}, \begin{pmatrix} x=2 \\ y=0 \end{pmatrix}, \begin{pmatrix} x=\frac{2}{3} \\ y=\frac{2}{3} \end{pmatrix} \Rightarrow \begin{array}{l} \boldsymbol{a}^{(1)} = (0,0) \\ \boldsymbol{a}^{(2)} = (0,2) \\ \boldsymbol{a}^{(3)} = (2,0) \\ \boldsymbol{a}^{(4)} = \left(\frac{2}{3},\frac{2}{3}\right) \end{array}$$

2° $\frac{\partial^2 f}{\partial x^2} = -2y$, $\frac{\partial^2 f}{\partial y^2} = -2x$, $\frac{\partial^2 f}{\partial x \partial y} = 2 - 2x - 2y$.

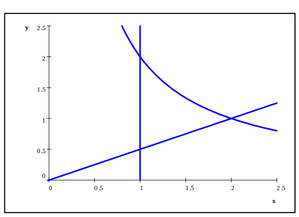
1. (a) $\boldsymbol{a}^{(1)} : \Delta_2 = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4$, $\Delta_1 = 0$, sedlo (nemá extrém)

$\boldsymbol{a}^{(2)} : \Delta_2 = \begin{vmatrix} -4 & -2 \\ -2 & 0 \end{vmatrix} = -4$, $\Delta_1 = -4$, sedlo (nemá extrém)

$\boldsymbol{a}^{(3)} : \Delta_2 = \begin{vmatrix} 0 & -2 \\ -2 & -4 \end{vmatrix} = -4$, $\Delta_1 = 0$, sedlo (nemá extrém)

$\boldsymbol{a}^{(4)} : \Delta_2 = \begin{vmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{vmatrix} = \frac{4}{3} > 0$, $\Delta_1 = -\frac{4}{3} < 0 \Rightarrow f\left(\frac{2}{3},\frac{2}{3}\right) = \frac{8}{27}$ lokálne maximum

3. Vypočítajte integrál $\iint_A \left(x^2+y\right) dxdy$, kde A je ohraničená krivkami : $y = \frac{1}{2}x$, $xy = 2$, $x = 1$. Načrtnite obrázok množiny A. **(20b)**



$2x = \frac{2}{x} \Leftrightarrow x^2 - 4 = 0 \Leftrightarrow (x-2)(x+2) = 0$

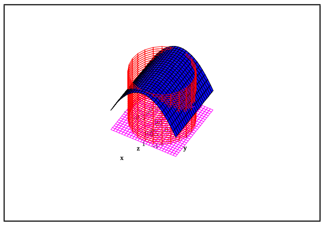
$$A: \begin{array}{ccccc} 1 & \le & x & \le & 2 \\ \frac{1}{2}x & \le & y & \le & \frac{2}{x} \end{array}$$

$\iint_A \left(x^2+y\right) dxdy = \int_1^2 \left(\int_{\frac{1}{2}x}^{\frac{2}{x}} \left(x^2+y\right) dy \right) dx =$

$= \int_1^2 \left(\left[x^2 y + \frac{y^2}{2} \right]_{\frac{1}{2}x}^{\frac{2}{x}} \right) dx = \int_1^2 \left(2x + \frac{2}{x^2} - \frac{1}{8}x^2 - \frac{1}{2}x^3 \right) dx =$

$= \left[-\frac{1}{8}x^4 - \frac{1}{24}x^3 + x^2 - \frac{2}{x} \right]_1^2 = \frac{11}{6}$

4. Použitím cylindrických súradníc vypočítajte objem množiny $A = \{(x,y,z)\,;\, x^2+y^2 \le 1,\, 0 \le z \le 2-y^2\}$. Načrtnite obrázok množiny A. (**20b**)



$x = r\cos\varphi, y = r\sin\varphi, z = z, J = r,$ potom z $x^2+y^2 \le 1,\, 0 \le z \le 2-y^2$ máme

$$A: \begin{array}{l} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z \le 2 - r^2\sin^2\varphi \end{array} \quad , E.O.\ [\varphi, r, z]:$$

$$V = \iiint\limits_{\widetilde{A}} 1dxdydz = \iiint\limits_{A} 1rd\varphi drdz =$$

$$= \left| A: \begin{array}{lll} x = r\cos\varphi, & 0 \le \varphi \le 2\pi, & \\ y = r\sin\varphi, & 0 \le r \le 1, & E.O.\,[\varphi, r, z] \\ z = z, J = r, & 0 \le z \le 2 - r^2\sin^2\varphi, & \end{array} \right| =$$

$$= \int_0^{2\pi} \left(\int_0^1 \left[\int_0^{2-r^2\sin^2\varphi} rdz \right] dy \right) dx = \int_0^{2\pi} \left(\int_0^1 \left[2r - r^3\sin^2\varphi \right] dr \right) d\varphi =$$

$$= \int_0^{2\pi} \left(\left[r^2 - \tfrac{r^4}{4}\sin^2\varphi \right]_0^1 \right) d\varphi = \int_0^{2\pi} \left(1^2 - \tfrac{1}{4}\sin^2\varphi \right) d\varphi =$$

$$= \int_0^{2\pi} d\varphi - \tfrac{1}{4}\int_0^{2\pi} \sin^2\varphi d\varphi = 2\pi - \tfrac{\pi}{4} = \tfrac{7}{4}\pi \ . \ (\mathbf{20b})$$