3. týždeň semestra

DÚ 1: Ukážte, že pre $n \in \mathbb{N}$ platí nasledujúci rekurentný vzťah:

$$I_n = \int (1-x^2)^n \, dx = \left(\frac{2n}{2n+1}\right) I_{n-1} + \frac{x(1-x^2)^n}{2n+1},$$

kde $I_0 = \int (1-x^2)^0 \, dx = \int 1 \, dx = x$.

Riešenie:

Pre $n \in \mathbb{N}$ máme:

$$\begin{aligned} I_n &= \int (1-x^2)^n \, dx = \int (1-x^2)^{n-1}(1-x^2) \, dx = \int (1-x^2)^{n-1} \, dx - \int x^2(1-x^2)^{n-1} \, dx \\ &= I_{n-1} - \int x\left(x(1-x^2)^{n-1}\right) dx = \left| \begin{matrix} f(x) = x & g'(x) = x(1-x^2)^{n-1} \\ f'(x) = 1 & g(x) = -\frac{1}{2n}(1-x^2)^n \end{matrix} \right| \\ &= I_{n-1} + \frac{x(1-x^2)^n}{2n} - \frac{1}{2n} \int (1-x^2)^n \, dx = I_{n-1} + \frac{x(1-x^2)^n}{2n} - \frac{1}{2n} I_n, \qquad (1) \end{aligned}$$

kde sme pri výpočte $g(x)$ využili, že:

$$\begin{aligned} \int x(1-x^2)^{n-1} \, dx &= -\frac{1}{2} \int (-2x)(1-x^2)^{n-1} \, dx = \left| \begin{matrix} s = 1-x^2 \\ ds = -2x \, dx \end{matrix} \right| = -\frac{1}{2} \int s^{n-1} \, ds = -\frac{1}{2n} s^n \\ &= -\frac{1}{2n}(1-x^2)^n \end{aligned}$$

Z (1) dostaneme:

$$I_n = I_{n-1} + \frac{x(1-x^2)^n}{2n} - \frac{1}{2n} I_n$$

$$\left(1 + \frac{1}{2n}\right) I_n = I_{n-1} + \frac{x(1-x^2)^n}{2n}$$

$$\left(\frac{2n+1}{2n}\right) I_n = I_{n-1} + \frac{x(1-x^2)^n}{2n}$$

$$I_n = \left(\frac{2n}{2n+1}\right) I_{n-1} + \frac{x(1-x^2)^n}{2n+1}$$

3. týždeň semestra

DÚ 2: Vypočítajte:

$$\int_2^3 \frac{\ln((2x-3)^2)}{\left(1-\frac{3}{2x}\right)}\,dx$$

Riešenie:

$$\int_2^3 \frac{\ln((2x-3)^2)}{\left(1-\frac{3}{2x}\right)}\,dx = \int_2^3 \frac{4x\ln(2x-3)}{(2x-3)}\,dx = \left| \begin{matrix} t = \ln(2x-3) & x = 3 \to t = \ln 3 & 2x-3 = e^t \\ dt = \dfrac{2}{2x-3}dx & x = 2 \to t = 0 & 2x = e^t + 3 \end{matrix} \right|$$

$$= \int_0^{\ln 3} t(e^t+3)\,dt = \left| \begin{matrix} f(t) = t & g'(t) = e^t + 3 \\ f'(t) = 1 & g(t) = e^t + 3t \end{matrix} \right| = [t(e^t+3t)]_0^{\ln 3} - \int_0^{\ln 3} e^t + 3t\,dt$$

$$= 3\ln 3 + 3\ln^2 3 - \left[e^t + \frac{3}{2}t^2\right]_0^{\ln 3} = 3\ln 3 + 3\ln^2 3 - 3 - \frac{3}{2}\ln^2 3 + 1$$

$$= \frac{3}{2}\ln^2 3 + 3\ln 3 - 2$$