10. týždeň semestra

DÚ: Vypočítajte:

$$\iint_I \frac{\mathbf{18x}}{\sqrt{\mathbf{2-2xy^2}}}\,\boldsymbol{dxdy},$$

ak $\boldsymbol{I = \left\langle \frac{1}{4}, \frac{1}{2} \right\rangle \times \langle 0, 1 \rangle}$**.**

Riešenie:

$$\iint_I \frac{18x}{\sqrt{2-2xy^2}}\,dxdy = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\int_0^1 \frac{18x}{\sqrt{2-2xy^2}}\,dy \right) dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\int_0^1 \frac{18x}{\sqrt{2}\sqrt{1-xy^2}}\,dy \right) dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{18\sqrt{x}}{\sqrt{2}} \left[\arcsin\left(\sqrt{x}y\right)\right]_0^1 dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{18\sqrt{x}}{\sqrt{2}} \arcsin\left(\sqrt{x}\right) dx = \left| \begin{matrix} s = \sqrt{x} \\ ds = \dfrac{dx}{2\sqrt{x}} \\ x = \dfrac{1}{4} \Rightarrow s = \dfrac{1}{2} \\ x = \dfrac{1}{2} \Rightarrow s = \dfrac{1}{\sqrt{2}} \end{matrix} \right|$$

$$= \frac{1}{\sqrt{2}} \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} 36s^2 \arcsin s\,ds = \left| \begin{matrix} u(s) = \arcsin s & v'(s) = 36s^2 \\ u'(s) = \dfrac{1}{\sqrt{1-s^2}} & v(s) = 12s^3 \end{matrix} \right|$$

$$= 6\sqrt{2}\left[s^3 \arcsin s\right]_{1/2}^{1/\sqrt{2}} - \frac{12}{\sqrt{2}} \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{s^3}{\sqrt{1-s^2}}\,ds = 3\frac{\pi}{4} - \frac{6\sqrt{2}}{8}\frac{\pi}{6} + \frac{12}{\sqrt{2}} \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{(-s)}{\sqrt{1-s^2}} s^2\,ds$$

$$= \left| \begin{matrix} t = \sqrt{1-s^2} \\ dt = \dfrac{-s}{\sqrt{1-s^2}}\,ds \\ s = \dfrac{1}{2} \Rightarrow t = \dfrac{\sqrt{3}}{2} \\ s = \dfrac{1}{\sqrt{2}} \Rightarrow t = \dfrac{1}{\sqrt{2}} \end{matrix} \right| = \frac{\pi}{8}\left(6-\sqrt{2}\right) + \frac{12}{\sqrt{2}} \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} 1 - t^2\,dt = \frac{\pi}{8}\left(6-\sqrt{2}\right) - \frac{12}{\sqrt{2}}\left[t - \frac{t^3}{3}\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{8}\left(6-\sqrt{2}\right) - \frac{12}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) + \frac{12}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) = \frac{\pi}{8}\left(6-\sqrt{2}\right) - \frac{9\sqrt{3}}{2\sqrt{2}} + 5$$

Iný postup:

$$\iint_I \frac{18x}{\sqrt{2-2xy^2}}\,dxdy = \int_{\frac{1}{4}}^{\frac{1}{2}}\left(\int_0^1 \frac{18x}{\sqrt{2-2xy^2}}\,dy\right)dx = \int_{\frac{1}{4}}^{\frac{1}{2}}\left(\int_0^1 \frac{18x}{\sqrt{2}\sqrt{1-xy^2}}\,dy\right)dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{18\sqrt{x}}{\sqrt{2}}\left[\arcsin\left(\sqrt{x}y\right)\right]_0^1 dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{18\sqrt{x}}{\sqrt{2}}\arcsin\left(\sqrt{x}\right) dx$$

$$= \begin{vmatrix} f(x) = \arcsin\left(\sqrt{x}\right) & g'(x) = \dfrac{18\sqrt{x}}{\sqrt{2}} \\ f'(x) = \dfrac{1}{2\sqrt{x}\sqrt{1-x}} & g(x) = 6\sqrt{2}x\sqrt{x} \end{vmatrix} = 6\sqrt{2}\left[x\sqrt{x}\arcsin\left(\sqrt{x}\right)\right]_{1/4}^{1/2} - \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{3\sqrt{2}x}{\sqrt{1-x}}dx$$

$$= \begin{vmatrix} f(x) = 3\sqrt{2}x & g'(x) = \dfrac{1}{\sqrt{1-x}} \\ f'(x) = 3\sqrt{2} & g(x) = -2\sqrt{1-x} \end{vmatrix} =$$

$$= 6\sqrt{2}\left(\frac{1}{2\sqrt{2}}\arcsin\frac{1}{\sqrt{2}} - \frac{1}{8}\arcsin\frac{1}{2}\right) + 6\sqrt{2}\left[x\sqrt{1-x}\right]_{\frac{1}{4}}^{\frac{1}{2}} - 6\sqrt{2}\int_{\frac{1}{4}}^{\frac{1}{2}} \sqrt{1-x}\,dx$$

$$= \frac{3}{4}\pi - \frac{6\sqrt{2}}{8}\frac{\pi}{6} + 6\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{8}\right) + 4\sqrt{2}\left[\left(\sqrt{1-x}\right)^3\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{\pi}{8}\left(6-\sqrt{2}\right) + 3 - \frac{3\sqrt{3}}{2\sqrt{2}} + 4\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \frac{3\sqrt{3}}{8}\right) = \frac{\pi}{8}\left(6-\sqrt{2}\right) + 3 - \frac{3\sqrt{3}}{2\sqrt{2}} + 2 - \frac{6\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\pi}{8}\left(6-\sqrt{2}\right) - \frac{9\sqrt{3}}{2\sqrt{2}} + 5$$