Príklad 1. [20]

Je daná funkcia

$$f(x,y) = \frac{x^2}{x^2+y^2}.$$

a, Vypočítajte rovnicu dotykovej roviny v bode $a = [1,3]$.

b, Vypočítajte deriváciu $\frac{\partial f}{\partial \vec{e}}(1,3)$, ak smer $\vec{e}$ je daný vektorom $\vec{u} = (1,1)$.

c, Vypočítajte limitu (ak existuje)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

$D_f = R^2 \setminus (0,0)$

$$f'_x = \frac{2x(x^2+y^2) - 2x^3}{(x^2+y^2)^2} \qquad f'_y = \frac{-2yx^2}{(x^2+y^2)^2}$$

a, Parc. der. sú spojité na D_f → f je diferencovateľná (na D_f), má dotyk. rovinu na D_f

rovnica dotyk. roviny v bode [1,3]

$$z - f(1,3) = f'_x(1,3)(x-1) + f'_y(1,3)(y-3)$$

$$f'_x(1,3) = \frac{2(1+9) - 2}{(1+9)^2} = \frac{18}{100} = \frac{9}{50}$$

$$f'_y(1,3) = \frac{-2\cdot 3\cdot 1^2}{(1+9)^2} = \frac{-6}{100} = \frac{-3}{50}$$

ve :

$$z - \frac{1}{10} = \frac{9}{50}(x-1) - \frac{3}{50}(y-3)$$

b, e daný $u = (1,1)$: $e = u \cdot \frac{1}{|u|}$

$$|u| = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

f je diferencovatelná $\to$ $\frac{\partial f}{\partial e}(1,3) = \text{grad} f(1,3) \cdot e$

$$= \left(\frac{9}{50}, \frac{-3}{50}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{9}{50\sqrt{2}} - \frac{3}{50\sqrt{2}} = \frac{6}{50\sqrt{2}}$$

$$= \frac{3}{25\sqrt{2}}$$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ $\qquad f(x,y) = \frac{x^2}{x^2+y^2}$

$\varphi_1(t) = (t,t) \qquad \varphi_1(t) \xrightarrow{t\to 0} (0,0)$

$\lim\limits_{t\to 0} f(\varphi_1(t)) = \lim\limits_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}$

$\frac{1}{2} \neq 0$

lim. neexistuje

$\varphi_2(t) = (0,t) \qquad \varphi_2(t) \xrightarrow{t\to 0} (0,0)$

$\lim\limits_{t\to 0} f(\varphi_2(t)) = \lim\limits_{t\to 0} \frac{0}{0+t^2} = 0$

Príklad 2. [20] Je daná funkcia

$$f(x,y) = 24xy - x^3 - 8y^3 + 5.$$

Nájdite jej lokálne extrémy.
Napíšte celý postup riešenia.

$D_f = R^2$

$f'_x = 24y - 3x^2 \qquad f'_y = 24x - 24y^2$

stac. body $\left.\begin{array}{l} 24y - 3x^2 = 0 \\ 24x - 24y^2 = 0 \end{array}\right\} \quad \begin{array}{l} 8y - x^2 = 0 \\ x = y^2 \end{array}$

$8y - y^4 = 0$

$y(8 - y^3) = 0$

$y(2-y)(4 + 2y + y^2) = 0$

korene $y = 0 \quad y = 2 \quad ; \quad 4 + 2y + y^2 = (y+1)^2 + 3 > 0$

stac. body: $(0,0) \quad (4,2)$

$f''_{xx} = -6x \quad f''_{xy} = 24 \quad f''_{yx} = 24 \quad f''_{yy} = -48y$

$M = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$

$M(0,0) = \begin{pmatrix} 0 & 24 \\ 24 & 0 \end{pmatrix} \qquad \det M_{0,0} = 0^2 - 24^2 < 0$

$(0,0)$ nie je lok. extrém

$M(4,2) = \begin{pmatrix} -24 & 24 \\ 24 & -96 \end{pmatrix} \qquad \det M_{4,2} = (-24)\cdot(-96) - 24^2 > 0$

$-24 < 0$

f má v $(4,2)$ OLMAX

$f(4,2) = 24\cdot 4\cdot 2 - 4^3 - 8\cdot 2^3 + 5 = 3\cdot 64 \; 64 - 64 + 5 = 69$

Príklad 3. [15]

Vypočítajte

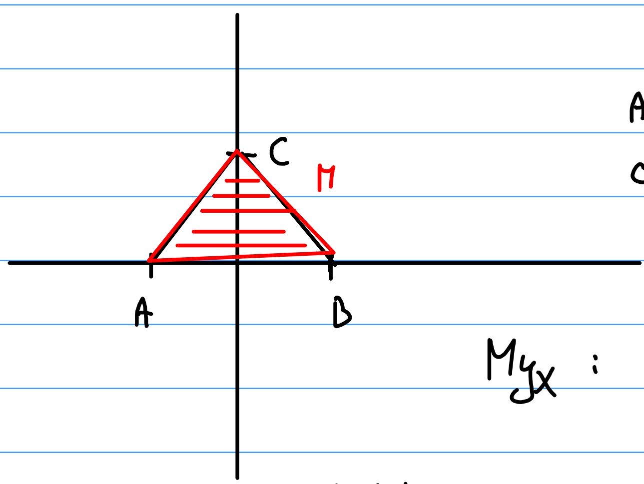
$$\iint_M e^y\,dxdy,$$

ak množina M je trojuholník ABC s vrcholmi $A=[-1,0]$, $B=[1,0]$, $C=[0,1]$.

Nakreslite množinu M.

Popíšte M ako elementárnu oblasť typu xy alebo yx.

Pri výpočte integrálu napíšte celý postup.



AC: $y = x - 1$

CB: $y = 1 - x$

M_{yx} : $0 \le y \le 1$

$y-1 \le x \le 1-y$

$$\iint_M e^x\,dxdy \overset{Fub}{=} \int_0^1\int_{y-1}^{1-y} e^y\,dx\,dy = \int_0^1 e^y\left[x\right]_{y-1}^{1-y}\,dy$$

$$= \int_0^1 e^y\left(1-y-(y-1)\right)\,dy = \int_0^1 2e^y - 2ye^y\,dy = (*)$$

$$2\int_0^1 ye^y\,dy = \begin{matrix} f=y & g'=e^y \\ f'=1 & g=e^y \end{matrix} = 2\left[ye^y\right]_0^1 - 2\int_0^1 e^y\,dy$$

$$\left[e^y\right]_0^1$$

$$= 2e - 2\cdot 0 - 2e + 2e^0 = 2$$

$$* = \left[2e^y\right]_0^1 - 2 = 2e - 2 - 2 = 2e - 4$$

Príklad 4. [15]

Použitím substitúcie vypočítajte

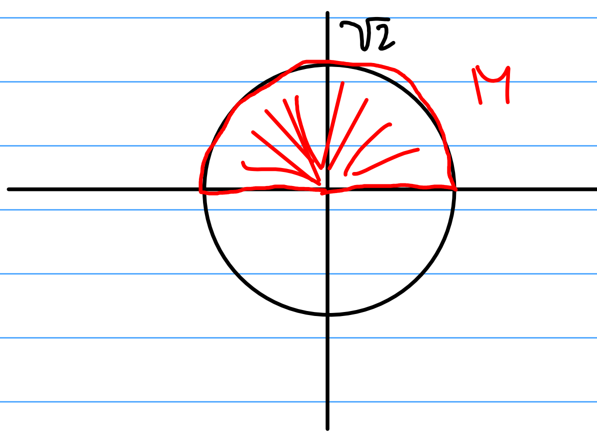
$$\iint_M \frac{1}{1+\sqrt{x^2+y^2}}\,dxdy,$$

ak množina M je daná nerovnosťami $x^2+y^2 \le 2$, $y \ge 0$.

Nakreslite množinu M.

Popíšte M ako elementárnu oblasť v polárnych súradniciach.

Pri výpočte integrálu napíšte celý postup.



$$M_{r\varphi}: \quad 0 \le r \le \sqrt{2}, \quad 0 \le \varphi \le \pi$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\det J(r,\varphi) = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & -r\cos\varphi \end{pmatrix} = r$$

$$\iint_M f(x,y)\,dx\,dy \overset{subst}{=} \iint_{M_{r\varphi}} f(r\cos\varphi, r\sin\varphi)\cdot r \; dr\,d\varphi$$

$$= \int_0^{\sqrt{2}}\int_0^{\pi} \frac{1}{1+r}\cdot r \; d\varphi\, dr = \int_0^{\sqrt{2}} \frac{r}{1+r}\left[\varphi\right]_0^{\pi} dr$$

$$= \pi\int_0^{\sqrt{2}} \frac{r}{1+r}\,dr = \pi\int_0^{\sqrt{2}} 1 - \frac{1}{1+r}\,dr$$

$$= \pi \left[r - \ln|1+r| \right]_0^{\sqrt{2}} = \pi \left(\sqrt{2} - \ln(1+\sqrt{2}) - 0 + \ln 1 \right)$$

$$= \pi \left(\sqrt{2} - \ln(1+\sqrt{2}) \right).$$