Domáce úlohy cvičenie 7 - riešenia

DÚ 1: Vypočítajte limitu elementárnymi úpravami:

$$\lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x+\pi)}$$

Riešenie:

$$\lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x+\pi)} = \text{"0/0"} = \lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x)\cos\pi + \cos(3x)\sin\pi}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \left(\frac{1-\cos x}{2}\right)}{-\sin(3x) + 0\cos(3x)} = -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{2\cos^2 x + \cos x - 1}{\sin(3x)}$$

$$= -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{\sin(2x+x)} = -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{\sin(2x)\cos x + \cos(2x)\sin x}$$

$$= -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{2\sin x\cos^2 x + (\cos^2 x - \sin^2 x)\sin x}$$

$$= -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{2\sin x\cos^2 x + (2\cos^2 x - 1)\sin x}$$

$$= -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{\sin x\,(4\cos^2 x - 1)} = -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{(2\cos x - 1)(\cos x + 1)}{\sin x\,(2\cos x - 1)(2\cos x + 1)}$$

$$= -\frac{1}{2}\lim_{x\to\frac{\pi}{3}} \frac{\cos x + 1}{\sin x\,(2\cos x + 1)} = -\frac{1}{2}\frac{\frac{1}{2}+1}{\frac{\sqrt{3}}{2}\left(2\frac{1}{2}+1\right)} = \frac{-3}{4\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

Iný postup:

$$\lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x+\pi)} = \text{"0/0"} = \left| \begin{array}{l} t = \frac{x}{2} \\ x \to \frac{\pi}{3} \\ t \to \frac{\pi}{6} \end{array} \right| = \lim_{t\to\frac{\pi}{6}} \frac{\cos^2(2t) - \sin^2 t}{\sin(6t+\pi)}$$

$$= \lim_{t\to\frac{\pi}{6}} \frac{(\cos^2 t - \sin^2 t)^2 - \sin^2 t}{\sin(6t)\cos\pi + \sin\pi\cos(6t)} = \lim_{t\to\frac{\pi}{6}} \frac{(1 - 2\sin^2 t)^2 - \sin^2 t}{-\sin(6t)}$$

$$= -\lim_{t\to\frac{\pi}{6}} \frac{1 - 4\sin^2 t + 4\sin^4 t - \sin^2 t}{\sin(4t+2t)}$$

$$= -\lim_{t\to\frac{\pi}{6}} \frac{1 - 5\sin^2 t + 4\sin^4 t}{\sin(4t)\cos(2t) + \sin(2t)\cos(4t)}$$

$$= -\lim_{t\to\frac{\pi}{6}} \frac{1 - 5\sin^2 t + 4\sin^4 t}{2\sin(2t)\cos^2(2t) + \sin(2t)\left(\cos^2(2t) - \sin^2(2t)\right)}$$

$$= -\lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{2\sin(2t)\cos^2(2t) + \sin(2t)\,(2\cos^2(2t)-1)} = -\lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{\sin(2t)\,(4\cos^2(2t)-1)}$$

$$= \left(\lim_{t\to\frac{\pi}{6}} \frac{-1}{\sin(2t)}\right)\left(\lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{4\cos^2(2t)-1}\right) = \frac{-1}{\frac{\sqrt{3}}{2}} \lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{4(\cos^2 t - \sin^2 t)^2 - 1}$$

$$= -\frac{2}{\sqrt{3}} \lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{4(1-2\sin^2 t)^2 - 1} = -\frac{2}{\sqrt{3}} \lim_{t\to\frac{\pi}{6}} \frac{1-5\sin^2 t + 4\sin^4 t}{3 - 16\sin^2 t + 16\sin^4 t}$$

$$= \left| \begin{array}{l} u = \sin^2 t \\ t \to \frac{\pi}{6} \\ u \to \frac{1}{4} \end{array} \right| = -\frac{2}{\sqrt{3}} \lim_{u\to\frac{1}{4}} \frac{1-5u+4u^2}{3-16u+16u^2} = -\frac{2}{\sqrt{3}} \lim_{u\to\frac{1}{4}} \frac{4\left(u-\frac{1}{4}\right)(u-1)}{16\left(u-\frac{1}{4}\right)\left(u-\frac{3}{4}\right)}$$

$$= \frac{-2}{4\sqrt{3}} \lim_{u\to\frac{1}{4}} \frac{u-1}{u-\frac{3}{4}} = \frac{-1}{2\sqrt{3}} \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{-3}{4\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

Ďalší postup:

$$\lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x+\pi)} = "0/0" = \lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \sin^2\left(\frac{x}{2}\right)}{\sin(3x)\cos\pi + \cos(3x)\sin\pi}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\cos^2 x - \left(\frac{1-\cos x}{2}\right)}{-\sin(3x) + 0\cos(3x)} = -\frac{1}{2} \lim_{x\to\frac{\pi}{3}} \frac{2\cos^2 x - 1 + \cos x}{\sin(3x)}$$

$$= -\frac{1}{2} \lim_{x\to\frac{\pi}{3}} \frac{\cos 2x + \cos x}{\sin\left(2\left(\frac{3}{2}x\right)\right)} = -\frac{1}{2} \lim_{x\to\frac{\pi}{3}} \frac{2\cos\left(\frac{2x+x}{2}\right)\cos\left(\frac{2x-x}{2}\right)}{2\sin\left(\frac{3}{2}x\right)\cos\left(\frac{3}{2}x\right)}$$

$$= -\frac{1}{2} \lim_{x\to\frac{\pi}{3}} \frac{\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos\left(\frac{3}{2}x\right)\sin\left(\frac{3}{2}x\right)} = -\frac{1}{2} \lim_{x\to\frac{\pi}{3}} \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{3}{2}x\right)} = -\frac{1}{2}\frac{\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{4}$$

DÚ 2: Vypočítajte limitu elementárnymi úpravami:

$$\boldsymbol{\lim_{x\to\infty} \frac{2e^x - e^{-x} + \cos x - 3\sin x}{5e^x + 4e^{-x} - 2\cos x - \sin x}}$$

Riešenie:

$$\lim_{x\to\infty} \frac{2e^x - e^{-x} + \cos x - 3\sin x}{5e^x + 4e^{-x} - 2\cos x - \sin x} = "\infty/\infty" = \lim_{x\to\infty} \frac{e^x\left(2 - e^{-2x} + \frac{\cos x - 3\sin x}{e^x}\right)}{e^x\left(5 + 4e^{-2x} - \frac{2\cos x + \sin x}{e^x}\right)}$$

$$= \lim_{x\to\infty} \frac{2 - e^{-2x} + \frac{\cos x - 3\sin x}{e^x}}{5 + 4e^{-2x} - \frac{2\cos x + \sin x}{e^x}} = \frac{\lim_{x\to\infty} 2 - e^{-2x} + \frac{\cos x - 3\sin x}{e^x}}{\lim_{x\to\infty} 5 + 4e^{-2x} - \frac{2\cos x + \sin x}{e^x}}$$

$$= \frac{2-0+0}{5+0-0} = \frac{2}{5},$$

kde sme využili, že $e^{-2x} \to 0$ pre $x \to \infty$ a rovnako aj $\frac{1}{e^x} \to 0$ pre $x \to \infty$. Keďže funkcie $(\cos x - 3\sin x)$ aj $(2\cos x + \sin x)$ sú ohraničené na celom $\mathbb{R}$, tak $\frac{\cos x - 3\sin x}{e^x} \to 0$ pre $x \to \infty$ a rovnako aj $\frac{2\cos x + \sin x}{e^x} \to 0$ pre $x \to \infty$.

DÚ 3: Vypočítajte limitu elementárnymi úpravami:

$$\lim_{x \to \frac{\pi}{2}+} \left(x^2 - \frac{\pi}{2}x\right) \operatorname{cotg}\left(x + \frac{\pi}{2}\right)$$

Riešenie:

$$\lim_{x \to \frac{\pi}{2}+} \left(x^2 - \frac{\pi}{2}x\right) \operatorname{cotg}\left(x + \frac{\pi}{2}\right) = \text{"}0.\infty\text{"} = \lim_{x \to \frac{\pi}{2}+} x\left(x - \frac{\pi}{2}\right) \operatorname{cotg}\left(x + \frac{\pi}{2}\right)$$

$$= \left(\lim_{x \to \frac{\pi}{2}+} x\right)\left(\lim_{x \to \frac{\pi}{2}+} \left(x - \frac{\pi}{2}\right) \operatorname{cotg}\left(x + \frac{\pi}{2}\right)\right) = \begin{vmatrix} t = x - \frac{\pi}{2} \\ x \to \frac{\pi}{2}+ \\ t \to 0+ \end{vmatrix}$$

$$= \frac{\pi}{2} \lim_{t \to 0+} t \operatorname{cotg}(t + \pi) = (\operatorname{cotg} \text{ je } \pi \text{ periodická funkcia}) = \frac{\pi}{2} \lim_{t \to 0+} t \operatorname{cotg} t$$

$$= \frac{\pi}{2} \lim_{t \to 0+} t \frac{\cos t}{\sin t} = \frac{\pi}{2} \lim_{t \to 0+} \frac{\cos t}{\frac{\sin t}{t}} = \frac{\pi}{2} \frac{\lim_{t \to 0+} \cos t}{\lim_{t \to 0+} \frac{\sin t}{t}} = \frac{\pi}{2} . \frac{1}{1} = \frac{\pi}{2}$$