Domáce úlohy cvičenie 1 - riešenia

Úloha 1: Nájdite $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$ také, že:

$$\boldsymbol{\frac{-y+ix}{1-2i}+\frac{x+iy}{2+3i}=1.}$$

Riešenie:

Upravme rovnicu zo zadania:

$$\frac{-y+ix}{1-2i}+\frac{x+iy}{2+3i}=1,$$

$$\left(\frac{-y+ix}{1-2i}\right)\left(\frac{1+2i}{1+2i}\right)+\left(\frac{x+iy}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)=1,$$

$$\frac{(-y+ix)(1+2i)}{1^2+(-2)^2}+\frac{(x+iy)(2-3i)}{2^2+3^2}=1,$$

$$\frac{-y+ix-2iy-2x}{1+4}+\frac{2x+2iy-3ix+3y}{4+9}=1,$$

$$\frac{-2x-y+i(x-2y)}{5}+\frac{2x+3y+i(2y-3x)}{13}=1,$$

$$-26x-13y+i(13x-26y)+10x+15y+i(10y-15x)=65,$$

$$-16x+2y+i(-2x-16y)=65.$$

Dve komplexné čísla sa rovnajú, ak sa rovnajú ich reálne aj imaginárne časti, teda musí platiť:

$$-16x+2y=65, \tag{1}$$

$$-2x-16y=0. \tag{2}$$

Z (2) máme:

$$x=-8y. \tag{3}$$

Dosadením (3) do (1) získame:

$$130y=65. \tag{4}$$

Zo (4) dostávame:

$$y=\frac{1}{2}. \tag{5}$$

Dosadením (5) za y v (3) máme:

$$x=-4.$$

Rovnica má teda jediné riešenie: $x=-4,\ y=\frac{1}{2}$.

Úloha 2: V obore komplexných čísel riešte rovnicu

$$\left(\boldsymbol{z}-\sqrt{\mathbf{3}}+\boldsymbol{i}\sqrt{\mathbf{2}}\right)^{\mathbf{4}}+\mathbf{4}=\mathbf{0}.$$

Riešenia vyjadrite v algebraickom tvare.

Riešenie:

Označme: $w = z - \sqrt{3} + i\sqrt{2}$. Potom: $w^4 + 4 = 0$ a teda $w^4 = -4$, čo je binomická rovnica. Vyriešme ju.

Za tým účelom zapíšme komplexné číslo $-4 + 0i$ v exponenciálnom tvare: $-4 = 4e^{i\pi}$. Táto binomická rovnica má štyri riešenia:

$$w_k = \sqrt{2}e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}, \text{kde } k = 0,1,2,3.$$

Teda:

$$w_0 = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i,$$

$$w_1 = \sqrt{2}e^{i\frac{3}{4}\pi} = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1 + i,$$

$$w_2 = \sqrt{2}e^{i\frac{5}{4}\pi} = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1 - i,$$

$$w_3 = \sqrt{2}e^{i\frac{7}{4}\pi} = \sqrt{2}\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i.$$

Keďže $z = w + \sqrt{3} - i\sqrt{2}$, tak riešeniami rovnice zo zadania sú:

$$z_0 = w_0 + \sqrt{3} - i\sqrt{2} = 1 + i + \sqrt{3} - i\sqrt{2} = \left(1 + \sqrt{3}\right) + i\left(1 - \sqrt{2}\right),$$

$$z_1 = w_1 + \sqrt{3} - i\sqrt{2} = -1 + i + \sqrt{3} - i\sqrt{2} = \left(-1 + \sqrt{3}\right) + i\left(1 - \sqrt{2}\right),$$

$$z_2 = w_2 + \sqrt{3} - i\sqrt{2} = -1 - i + \sqrt{3} - i\sqrt{2} = \left(-1 + \sqrt{3}\right) + i\left(-1 - \sqrt{2}\right),$$

$$z_3 = w_3 + \sqrt{3} - i\sqrt{2} = 1 - i + \sqrt{3} - i\sqrt{2} = \left(1 + \sqrt{3}\right) + i\left(-1 - \sqrt{2}\right).$$