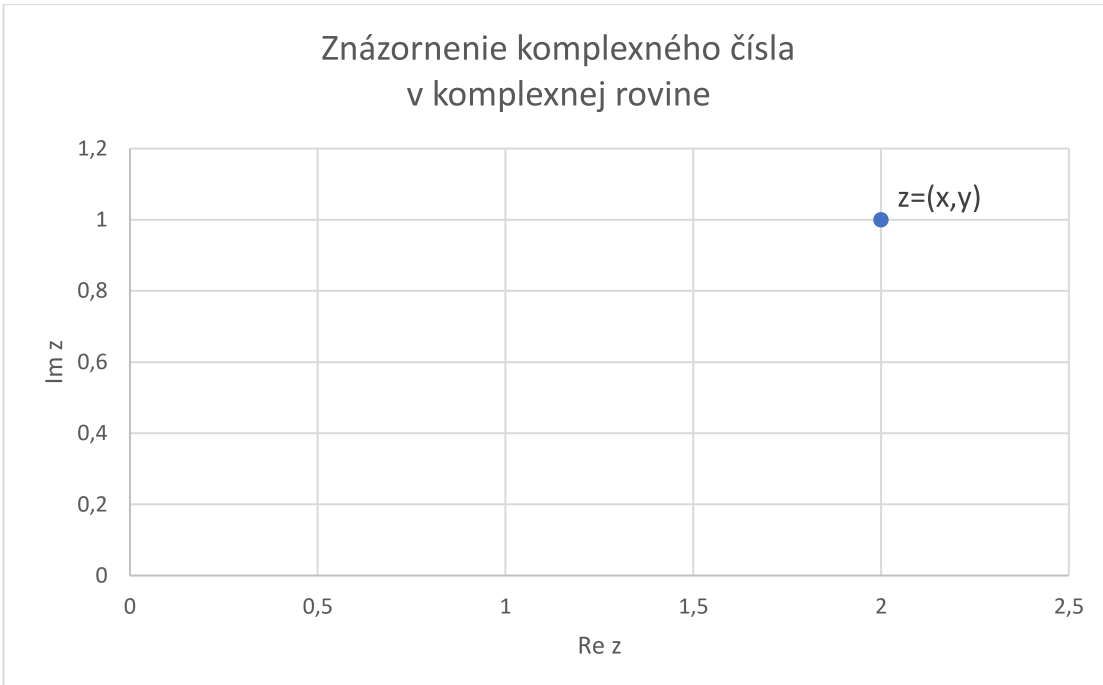
KOMPLEXNÉ ČÍSLA

Algebraický tvar komplexného čísla $z \in \mathbb{C} : z = x + iy$, kde $x \in \mathbb{R}$, $y \in \mathbb{R}$ a $i^2 = -1$.
Reálna časť komplexného čísla $z \in \mathbb{C} : x = \text{Re}\, z$.
Imaginárna časť komplexného čísla $z \in \mathbb{C} : y = \text{Im}\, z$.



Sčitovanie komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde $z_1 = x_1 + iy_1$ a $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Príklad 1: Sčítajte komplexné čísla $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde:

a) $z_1 = 3 + 2i$, $z_2 = -1 - 5i$

$$z_1 + z_2 = (3 + 2i) + (-1 - 5i) = (3 - 1) + i(2 - 5) = 2 - 3i$$

b) $z_1 = -2 + 4i$, $z_2 = 2 - 4i$

$$z_1 + z_2 = (-2 + 4i) + (2 - 4i) = (2 - 2) + i(4 - 4) = 0 + 0i = 0$$

c) $z_1 = -\sqrt{2} + \frac{5}{13}i$, $z_2 = 1 - \frac{7}{26}i$

$$z_1 + z_2 = \left(-\sqrt{2} + \frac{5}{13}i\right) + \left(1 - \frac{7}{26}i\right) = \left(1 - \sqrt{2}\right) + i\left(\frac{10}{26} - \frac{7}{26}\right) = 1 - \sqrt{2} + \frac{3}{26}i$$

d) $z_1 = -3 - \frac{5}{6}i,\ z_2 = -3 + \frac{5}{6}i$

$$z_1 + z_2 = \left(-3 - \frac{5}{6}i\right) + \left(-3 + \frac{5}{6}i\right) = (-3-3) + i\left(-\frac{5}{6} + \frac{5}{6}\right) = -6 + 0i = -6$$

Násobenie komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde $z_1 = x_1 + iy_1$ a $z_2 = x_2 + iy_2$:

$$\begin{aligned} z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \end{aligned}$$

Príklad 2: Vynásobte komplexné čísla $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde:

a) $z_1 = 3 + 2i,\ z_2 = 1 - 5i$

$$z_1z_2 = (3+2i)(1-5i) = 3 - 15i + 2i - 10i^2 = 3 + 10 + i(2-15) = 13 - 13i$$

b) $z_1 = 13,\ z_2 = 1 - i$

$$z_1z_2 = 13(1-i) = 13 - 13i$$

c) $z_1 = 8i,\ z_2 = -3 + \frac{1}{4}i$

$$z_1z_2 = 8i\left(-3 + \frac{1}{4}i\right) = -24i + \frac{8}{4}i^2 = -2 - 24i$$

d) $z_1 = -3 - \frac{5}{6}i,\ z_2 = -3 + \frac{5}{6}i$

$$z_1z_2 = \left(-3 - \frac{5}{6}i\right)\left(-3 + \frac{5}{6}i\right) = (-3)^2 - \left(\frac{5}{6}i\right)^2 = 9 - \frac{25}{36}i^2 = 9 + \frac{25}{36} = \frac{324}{36} + \frac{25}{36} = \frac{349}{36}$$

Špeciálne:

$$\begin{gathered} i^2 = -1, \\ i^3 = i^2 i = -i, \\ i^4 = i^3 i = -i^2 = 1, \\ i^5 = i^4 i = i, \\ i^6 = -1, \\ i^7 = -i, \\ i^8 = 1, \\ i^9 = i, \\ \vdots \end{gathered}$$

Príklad 3: Vypočítajte i^{3259}.

$$i^{3259} = i^{3256+3} = i^{814.4+3} = (i^4)^{814} i^3 = 1.(-i) = -i$$

Odčitovanie komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde $z_1 = x_1 + iy_1$ a $z_2 = x_2 + iy_2$:

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 + iy_1) + (-x_2 - iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Komplexne združené číslo ku komplexnému číslu $z = x + iy$ je číslo $\bar{z} = x - iy$.
Platí:

$$z + \bar{z} = 2x = 2\text{Re}\, z \Longrightarrow \text{Re}\, z = \frac{z + \bar{z}}{2},$$
$$z - \bar{z} = 2iy = 2i\text{Im}\, z \Longrightarrow \text{Im}\, z = \frac{z - \bar{z}}{2i},$$
$$z\bar{z} = (x + iy\,)(x - iy) = x^2 + y^2.$$

Delenie komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde $z_1 = x_1 + iy_1$ a $z_2 = x_2 + iy_2 \neq 0$:

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}\frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{x_1x_2 - ix_1y_2 + iy_1x_2 + y_1y_2}{x_2^2 + y_2^2}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Príklad 4: Vydeľte komplexné čísla $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, kde:

a) $z_1 = 2 - i$, $z_2 = -1 + 3i$

$$\frac{z_1}{z_2} = \frac{2 - i}{-1 + 3i} = \left(\frac{2 - i}{-1 + 3i}\right)\left(\frac{-1 - 3i}{-1 - 3i}\right) = \frac{(2 - i)(-1 - 3i)}{(-1)^2 + 3^2} = \frac{-2 - 3 - 6i + i}{1 + 9} = \frac{-5 - 5i}{10}$$
$$= -\frac{1}{2} - \frac{1}{2}i$$

b) $z_1 = 13$, $z_2 = 1 - i$

$$\frac{z_1}{z_2} = \frac{13}{1 - i} = \left(\frac{13}{1 - i}\right)\left(\frac{1 + i}{1 + i}\right) = \frac{13 + 13i}{2} = \frac{13}{2} + \frac{13}{2}i$$

c) $z_1 = 2 + 8i$, $z_2 = i$

$$\frac{z_1}{z_2} = \frac{2 + 8i}{i} = \left(\frac{2 + 8i}{i}\right)\left(\frac{-i}{-i}\right) = \frac{-2i - 8i^2}{-i^2} = \frac{-2i + 8}{1} = 8 - 2i$$

d) $z_1 = -18 - 9i$, $z_2 = 3 + 6i$

$$\frac{z_1}{z_2} = \frac{-18 - 9i}{3 + 6i} = \frac{-9(2 + i)}{3(1 + 2i)} = \frac{-3(2 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{-3(2 - 4i + i + 2)}{1 + 4} = \frac{-3(4 - 3i)}{5}$$
$$= -\frac{12}{5} + \frac{9}{5}i$$

Goniometrický (trigonometrický) tvar komplexného čísla $z \in \mathbb{C} : z = |z|(\cos\varphi + i\sin\varphi)$, kde $|z| = \sqrt{x^2 + y^2}$ je absolútna hodnota komplexného čísla $z = x + iy$ a ak $z \neq 0$, tak orientovaný uhol φ, pre ktorý $z = |z|(\cos\varphi + i\sin\varphi)$, nazývame argument komplexného čísla z.
Uhol $\varphi \in \langle 0, 2\pi)$ (alebo $\varphi \in \langle -\pi, \pi)$) je teda taký, že: $\cos\varphi = \frac{x}{|z|}$ a súčasne $\sin\varphi = \frac{y}{|z|}$.
Keďže $z\bar{z} = x^2 + y^2$, tak $|z| = \sqrt{z\bar{z}}$.

Exponenciálny tvar komplexného čísla $z \in \mathbb{C} : z = |z|e^{i\varphi}$.
Eulerova formula: $e^{i\varphi} = \cos\varphi + i\sin\varphi$.

Príklad 5: Komplexné číslo $z \in \mathbb{C}$ zapíšte v goniometrickom a exponenciálnom tvare, ak:

a) $z = 1 - i\sqrt{3}$

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2,$$

$$\varphi \in \langle 0, 2\pi) \text{ je taký, že: } \cos\varphi = \frac{1}{2} \text{ a súčasne } \sin\varphi = \frac{-\sqrt{3}}{2} \Longrightarrow \varphi = 2\pi - \frac{\pi}{3} = \frac{5}{3}\pi$$

$$\text{Teda: } z = 2\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right), \text{ resp. } z = 2e^{i\frac{5}{3}\pi}.$$

b) $z = -1 - i$

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2},$$

$$\varphi \in \langle 0, 2\pi) \text{ je taký, že: } \cos\varphi = -\frac{1}{\sqrt{2}} \text{ a súčasne } \sin\varphi = -\frac{1}{\sqrt{2}} \Longrightarrow \varphi = \pi + \frac{\pi}{4} = \frac{5}{4}\pi$$

$$\text{Teda: } z = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right), \text{ resp. } z = \sqrt{2}e^{i\frac{5}{4}\pi}.$$

c) $z = \sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}$

$$|z| = \sqrt{\sqrt{2} + 1 + \sqrt{2} - 1} = \sqrt{2\sqrt{2}},$$

$$\varphi \in \langle 0, 2\pi) \text{ je taký, že: } \cos\varphi = \frac{\sqrt{\sqrt{2}+1}}{\sqrt{2\sqrt{2}}} \text{ a súčasne } \sin\varphi = \frac{\sqrt{\sqrt{2}-1}}{\sqrt{2\sqrt{2}}} \Longrightarrow \varphi \in \left(0, \frac{\pi}{2}\right)$$

$$\left.\begin{aligned} \cos\varphi &= \frac{\sqrt{\sqrt{2}+1}}{\sqrt{2\sqrt{2}}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} \\ \sin\varphi &= \frac{\sqrt{\sqrt{2}-1}}{\sqrt{2\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} \end{aligned}\right\} \Longrightarrow \varphi = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

$$\text{Teda: } z = 2^{3/4}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right), \text{ resp. } z = 2^{3/4}e^{i\frac{\pi}{8}}.$$

Súčin komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$ v goniometrickom tvare: $z_1 = |z_1|(\cos\varphi + i\sin\varphi)$, $z_2 = |z_2|(\cos\psi + i\sin\psi)$:

$$\begin{aligned} z_1 z_2 = |z_1|(\cos\varphi + i\sin\varphi)|z_2|(\cos\psi + i\sin\psi) &= |z_1||z_2|(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) \\ &= |z_1||z_2|(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi)) \\ &= |z_1||z_2|(\cos(\varphi+\psi) + i\sin(\varphi+\psi)) \end{aligned}$$

Podiel komplexných čísel $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$ v goniometrickom tvare: $z_1 = |z_1|(\cos\varphi + i\sin\varphi)$, $z_2 = |z_2|(\cos\psi + i\sin\psi) \neq 0$:

$$
\begin{aligned}
\frac{z_1}{z_2} = \frac{|z_1|(\cos\varphi + i\sin\varphi)}{|z_2|(\cos\psi + i\sin\psi)} &= \frac{|z_1|(\cos\varphi + i\sin\varphi)(\cos\psi - i\sin\psi)}{|z_2|(\cos\psi + i\sin\psi)(\cos\psi - i\sin\psi)} \\
&= \frac{|z_1|(\cos\varphi\cos\psi + \sin\varphi\sin\psi + i(\sin\varphi\cos\psi - \cos\varphi\sin\psi))}{|z_2|(\cos^2\psi + \sin^2\psi)} \\
&= \frac{|z_1|}{|z_2|}(\cos(\varphi - \psi) + i\sin(\varphi - \psi))
\end{aligned}
$$

Moivreova formula: $(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$ pre všetky $n \in \mathbb{N}$.
Umocňovanie komplexného čísla $z \in \mathbb{C}$: $z^n = \big(|z|(\cos\varphi + i\sin\varphi)\big)^n = |z|^n(\cos\varphi + i\sin(n\varphi))^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$, kde $n \in \mathbb{N}$.

Príklad 6: Vypočítajte z^n, ak:

a) $z = 1 - i\sqrt{3}, n = 3$

Z **príkladu 5a)** vieme, že: $z = 2\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right)$.
Potom:

$$
\begin{aligned}
z^3 = 2^3\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right)^3 &= 8\left(\cos\left(3\frac{5}{3}\pi\right) + i\sin\left(3\frac{5}{3}\pi\right)\right) \\
&= 8(\cos(5\pi) + i\sin(5\pi)) = 8(\cos(\pi + 4\pi) + i\sin(\pi + 4\pi)) \\
&= 8(\cos\pi + i\sin\pi) = 8(-1 + 0i) = -8
\end{aligned}
$$

b) $z = -1 - i, n = 5$

Z **príkladu 5b)** vieme, že: $z = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$.
Potom:

$$
\begin{aligned}
z^5 = \left(\sqrt{2}\right)^5\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)^5 &= 2^{5/2}\left(\cos\left(\frac{25}{4}\pi\right) + i\sin\left(\frac{25}{4}\pi\right)\right) \\
&= 2^{5/2}\left(\cos\left(\frac{\pi}{4} + 6\pi\right) + i\sin\left(\frac{\pi}{4} + 6\pi\right)\right) = 2^{5/2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\
&= 4\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 4 + 4i
\end{aligned}
$$

Príklad 7: Vypočítajte $\left(\frac{-\sqrt{3}+i}{1-i}\right)^{12}$.

Exponenciálny tvar komplexného čísla $-\sqrt{3} + i$ je $2e^{i\frac{5}{6}\pi}$ a komplexného čísla $1 - i$ je $\sqrt{2}e^{i\frac{7}{4}\pi}$.
Preto podiel: $\frac{-\sqrt{3}+i}{1-i} = \frac{2e^{i\frac{5}{6}\pi}}{\sqrt{2}e^{i\frac{7}{4}\pi}} = \sqrt{2}e^{i\left(\frac{5}{6}\pi - \frac{7}{4}\pi\right)} = \sqrt{2}e^{-i\frac{11}{12}\pi} = \sqrt{2}e^{i\frac{13}{12}\pi}$. Z toho:

$$\left(\frac{-\sqrt{3}+i}{1-i}\right)^{12} = \left(\sqrt{2}e^{i\frac{13}{12}\pi}\right)^{12} = 2^6 e^{i13\pi} = 64e^{i(\pi+12\pi)} = 64e^{i\pi} = 64(\cos\pi + i\sin\pi) = -64$$

Riešenia binomickej rovnice $z^n = w$, kde $z \in \mathbb{C}$, $w \in \mathbb{C}$ a $n \in \mathbb{N}$. Ak goniometrický, resp. exponenciálny tvar komplexného čísla w je $w = |w|(\cos\psi + i\sin\psi)$, resp. $w = |w|e^{i\psi}$, tak všetky riešenia binomickej rovnice $z^n = w$ sú:

$$z_k = \sqrt[n]{|w|}\left(\cos\left(\frac{\psi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\psi}{n} + k\frac{2\pi}{n}\right)\right), \qquad \text{kde } k = 0,1,2,\dots,n-1$$

resp.:

$$z_k = \sqrt[n]{|w|}e^{i\left(\frac{\psi}{n}+k\frac{2\pi}{n}\right)}, \qquad \text{kde } k = 0,1,2,\dots,n-1$$

Príklad 8: Nájdite všetky riešenia rovnice $z^n = w$, ak:

a) $w = 8i, n = 3$

Exponenciálny tvar komplexného čísla w je $w = 8e^{i\frac{\pi}{2}}$, tj. $|w| = 8$ a $\psi = \frac{\pi}{2}$.

Riešenia rovnice teda sú:

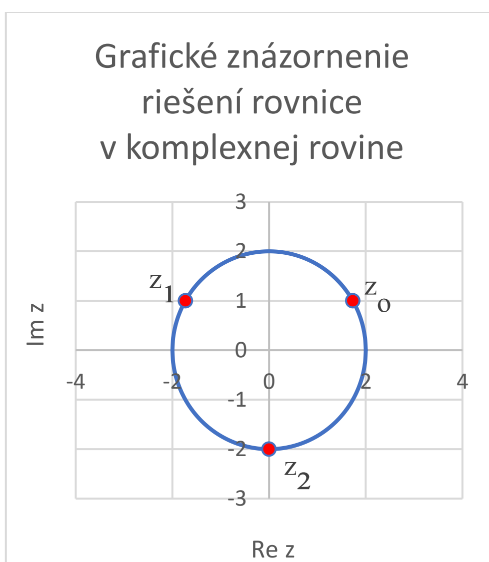
$$z_k = \sqrt[3]{8}e^{i\left(\frac{\pi}{6}+k\frac{2\pi}{3}\right)}, \text{kde } k = 0,1,2,$$

tj.:

$$z_0 = 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{3} + i,$$

$$z_1 = 2e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = 2e^{i\frac{5}{6}\pi} = 2\left(\cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\sqrt{3} + i,$$

$$z_2 = 2e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = 2e^{i\frac{3}{2}\pi} = 2\left(\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)\right) = 2(0 - i) = -2i.$$



b) $w = -4, n = 4$

Exponenciálny tvar komplexného čísla w je $w = 4e^{i\pi}$, tj. $|w| = 4$ a $\psi = \pi$.

Riešenia rovnice teda sú:

$$z_k = \sqrt[4]{4}e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}, \text{kde } k = 0,1,2,3,$$

tj.:

$$z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1 + i,$$

$$z_1 = \sqrt{2}e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = \sqrt{2}e^{i\frac{3}{4}\pi} = \sqrt{2}\left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -1 + i,$$

$$z_2 = \sqrt{2}e^{i\left(\frac{\pi}{4}+\pi\right)} = \sqrt{2}e^{i\frac{5}{4}\pi} = \sqrt{2}\left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -1 - i,$$

$$z_3 = \sqrt{2}e^{i\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)} = \sqrt{2}e^{i\frac{7}{4}\pi} = \sqrt{2}\left(\cos\left(\frac{7}{4}\pi\right) + i\sin\left(\frac{7}{4}\pi\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 1 - i.$$