Skúška z matematiky 1 (B)

7. januára 2009

Pretože ostatné skupiny mali približne rovnakú skúšku uvádzam iba riešenie skupiny B

Príklady:

1. Rozložte racionálnu funkciu

$$\frac{6x^3 - 8x^2 + x - 8}{(x^2 + x + 1)(x - 1)^2}$$

nad $\boldsymbol{R}$ na elementárne zlomky. (10 bodov)

Riešenie: $\frac{6x^3-8x^2+x-8}{(x^2+x+1)(x-1)^2} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{cx+d}{(x^2+x+1)}$

$\frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{cx+d}{(x^2+x+1)} = \frac{(a+c)x^3+(b+d-2c)x^2+(b+c-2d)x+(-a+b+d)}{(x^2+x+1)(x-1)^2} \Longrightarrow$

$$\begin{array}{rrrrl} a & & +c & & = 6 \\ & b & -2c & +d & = -8 \\ & b & +c & -2d & = 1 \\ -a & +b & & +d & = -8 \end{array} \Longrightarrow$$

$\frac{6x^3-8x^2+x-8}{(x^2+x+1)(x-1)^2} = \frac{4}{(x-1)} - \frac{3}{(x-1)^2} + \frac{2x-1}{x+x^2+1}.$

2. Vypočítajte $\boldsymbol{A}^{-1}$ a $\det(\boldsymbol{A})$, ak $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 & -1 \\ 0 & -1 & 1 & -2 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. (10 bodov)

Riešenie: $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{\sim}{\overset{r_1:=:r_3}{r_2:=:r_4}} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 & 0 & 0 \end{pmatrix} \underset{\sim}{r_3:=:r_3-2r_1}$

$\underset{\sim}{r_3:=:r_3-2r_1} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & -3 & 1 & 0 & -2 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 & 0 & 0 \end{pmatrix} \underset{\sim}{\overset{r_1:=:r_1-2r_2}{\overset{r_3:=:r_3+2r_2}{r_4:=:r_4+r_2}}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 & 1 \end{pmatrix} \underset{\sim}{r_4:=:r_4-r_3}$

$\underset{\sim}{r_4:=:r_4-r_3} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & -1 \end{pmatrix} \underset{\sim}{\overset{r_1:=:r_1-r_4}{r_3:=:r_3+3r_4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 3 & 4 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & -1 \end{pmatrix}$

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & -1 \end{pmatrix}$$

$\det(\boldsymbol{A}) = \begin{vmatrix} 2 & 2 & 1 & -1 \\ 0 & -1 & 1 & -2 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1.(-1)^{4+2} \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{vmatrix} = 2-2+0-(-1+0+0) =$
1.

3. (a) Vypočítajte limitu

$$\lim_{x\to\infty} \frac{\ln\left(x^2-x+1\right)}{\ln\left(x^{10}+x+1\right)}.$$

(5 bodov)

(b) Zistite, či rad

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$$

konverguje alebo diverguje. (5 bodov)

Riešenie: **a.** $\lim_{x\to\infty} \frac{\ln\left(x^2-x+1\right)}{\ln(x^{10}+x+1)} \overset{\infty}{\underset{\infty}{=}} \lim_{x\to\infty} \frac{\frac{2x-1}{x^2-x+1}}{\frac{10x^9-1}{x^{10}+x+1}} = \lim_{x\to\infty} \frac{(2x-1)\left(x^{10}+x+1\right)}{(x^2-x+1)(10x^9-1)} =$
$\lim_{x\to\infty} \frac{x^{11}\left(2-\frac{1}{x}\right)\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)}{x^{11}\left(1-\frac{1}{x}+\frac{1}{x^2}\right)\left(10-\frac{1}{x^9}\right)} = \frac{1}{5}$

b. $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$, podielové kritérium $a_n = \frac{2n-1}{2^n}$, $a_{n+1} = \frac{2n+1}{2^{n+1}}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2n+1}{2^{n+1}}}{\frac{2n-1}{2^n}} = \lim_{n\to\infty} \frac{(2n+1)2^n}{(2n-1)2^{n+1}} = \frac{1}{2}$ rad konverguje

4. Zistite priebeh funkcie

$$f(x) = (1-4x)\,e^{4x}$$

a načrtnite jej graf. (10 bodov)

Riešenie:

1° $f(x) = (1-4x)\,e^{4x}$, $D(f) = \mathbf{R}$, spojitá, $f\left(\frac{1}{4}\right) = 0$, nemá a.b.s.

2° $x \in D(f) \Longrightarrow -x \in D(f)$, $f(-x) = (1+4x)\,e^{-4x} \neq \begin{cases} f(x) \\ -f(x) \end{cases} \Longrightarrow f$ nie je párna, f nie je nepárna

3° Funkcia f nie je periodická (jeden n.b.)

4°-5° $f'(x) = \frac{d}{dx}\left((1-4x)\,e^{4x}\right) = -16xe^{4x}$
pre $x \in (-\infty, 0) \Rightarrow f'(x) > 0 \Rightarrow f \nearrow$
pre $x = 0$, $\max f(x) = f(0) = 1$,
pre $(0,\infty) \Rightarrow f'(x) < 0 \Rightarrow f \searrow$

6°-7° $f''(x) = \frac{d}{dx}\left((-16x)\,e^{4x}\right) = (-16)(1+4x)\,e^{4x}$
pre $x \in \left(-\infty, -\frac{1}{4}\right) \Rightarrow f''(x) > 0 \Rightarrow f\cup$,
v $x = -\frac{1}{4}$ má f inflexný bod, $f\left(-\frac{1}{4}\right) = 2e^{-1}$
pre $x \in \left(-\frac{1}{4}, \infty\right) \Rightarrow f''(x) < 0 \Rightarrow f\cap$.

8° $\lim_{x\longrightarrow-\infty} f(x) = \lim_{x\longrightarrow-\infty} \frac{(1-4x)}{e^{-4x}} = \lim_{x\longrightarrow-\infty} \frac{-4}{-4e^{-4x}} = 0$, $\lim_{x\longrightarrow\infty} f(x) = -\infty$. Asymptota v $-\infty : y = 0$, Asymptota v ∞ : $k = \lim_{x\to\infty} \frac{(1-4x)e^{4x}}{x} = \lim_{x\to\infty} \frac{-4e^{4x}+4(1-4x)e^{4x}}{1} = -\infty \Longrightarrow$asymptotu v ∞ nemá.

$H(f) = (-\infty, 1\rangle$.

Graf