13.50

1. [6 bodov] Riešte rovnicu $x^4 = -4$.
   Výsledok znázornite, vyjadrite v algebraickom tvare a polynóm $f(x) = x^4 + 4$ rozložte na súčin ireducibilných polynómov nad C.
2. [6 bodov] Racionálnu funkciu $r(x) = \dfrac{x^3 + x^2 + 4x - 4}{(x^2 + 2x + 5)^2}$ napíšte ako súčet elementárnych zlomkov nad R.
3. [6 bodov] Riešte sústavu
$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 0 \\ -x_1 + x_2 + 2x_3 + x_4 &= 3 \\ 3x_1 - x_2 + 4x_3 - x_4 &= 3 \end{aligned}$$
4. [2 body] Zistite, či je číslo $c = -2$ koreňom polynómu $f(x) = -3x^4 + x^3 + 12x^2 + 8$.

---

1. $x^4 = 4(\cos\pi + i\sin\pi)$
   $x_k = \sqrt{2}\Big[\cos\big(\frac{\pi}{4} + k\frac{\pi}{2}\big) + i\sin\big(\frac{\pi}{4} + k\frac{\pi}{2}\big),\ k = 0, 1, 2, 3$
   $x_0 = 1 + i$
   $x_1 = -1 + i$
   $x_2 = -1 - i$
   $x_3 = 1 - i$
   $\underline{f(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) = (x - 1 - i)(x + 1 - i)(x + 1 + i)(x - 1 + i)},$
   čísla $\pm 1 \pm i$ znázorniť v komplexnej rovine.
2. $D = 4 - 4 \cdot 5 = -16 < 0$
$$\frac{x^3 + x^2 + 4x - 4}{(x^2 + 2x + 5)^2} = \frac{Ax + B}{(x^2 + 2x + 5)^2} + \frac{Cx + D}{x^2 + 2x + 5}$$
$$\begin{aligned} x^3 + x^2 + 4x - 4 &= Ax + B + (Cx + D)(x^2 + 2X + 5) \\ &= Ax + B + Cx^3 + 2Cx^2 + 5Cx + Dx^2 + 2Dx + 5D \\ &= Cx^3 + (2C + D)x^2 + (A + 5C + 2D)x + (B + 5D) \end{aligned}$$
$$\Longrightarrow\ C = 1$$
$$2C + D = 2 + D = 1 \ \Longrightarrow\ D = -1$$
$$2D + 5C + A = -2 + 5 + A = 4 \ \Longrightarrow\ A = 1$$
$$B + 5D = B - 5 = -4 \ \Longrightarrow\ B = 1$$
$$r(x) = \underline{\frac{x + 1}{(x^2 + 2x + 5)^2} + \frac{x - 1}{x^2 + 2x + 5}}$$
3.
$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ -1 & 1 & 2 & 1 & 3 \\ 3 & -1 & 4 & -1 & 3 \end{array}\right)_{A_{1*} \leftrightarrow A_{2*}} \sim \left(\begin{array}{cccc|c} -1 & 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & -1 & 0 \\ 3 & -1 & 4 & -1 & 3 \end{array}\right)_{\substack{A_{2*} + 2A_{1*}, \\ A_{3*} + 3A_{1*}, A_{1*} \cdot (-1)}}$$
$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 0 & 1 & 5 & 1 & 6 \\ 0 & 2 & 10 & 2 & 12 \end{array}\right)_{A_{3*} - 2A_{2*}} \sim \left(\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 0 & 1 & 5 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)_{A_{1*} + A_{2*}}$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 3 \\ 0 & 1 & 5 & 1 & 6 \end{array}\right) \qquad \mathcal{R} = \underline{\{(3 - 3a, 6 - 5a - b, a, b) \colon a, b \in R\}}\,.$$
4.

	-3	1	12	0	8
		6	-14	4	-8
-2	-3	7	-2	4	$0 = f(c)$

$\Longrightarrow\ c = -2$ je koreň polynómu f.

13.00

1. [6 bodov] Riešte sústavu rovníc:
$$\begin{aligned} (1-i)x_1 + (1-2i)x_2 + x_3 &= 2-i \\ (2-i)x_2 + ix_3 &= 1 \\ (1+i)x_3 &= 1-i \end{aligned}$$

2. [6 bodov] Napíšte kanonický rozklad nad R aj nad C polynómu $f(x) = x^5 + 2x^4 + 2x^3 - 4x^2 - 11x + 10$.

3. [6 bodov] Vypočítajte $A \cdot B^{-1}$ pre matice $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$.

4. [2 body] Napíšte stupeň polynómu $f(x) = (x^2 + 2x + 3)^2 (2x-1)^3$.

1. $(1+i)x_3 = 1-i \implies x_3 = \frac{1-i}{1+i}\frac{1-i}{1-i} = \frac{1-2i-1}{2} = -i$,
$(2-i)x_2 + i \cdot (-i) = 1 \implies (2-i)x_2 + 1 = 1 \implies x_2 = 0$
$(1-i)x_1 + (1-2i)x_2 + x_3 = (1-i)x_1 - i = 2-i \implies x_1 = \frac{2}{1-i} = 1+i$
$\underline{\mathcal{R} = \{(1+i, 0, -i)\}}$

2. $p|10 \implies p \in \{\pm 1, \pm 2, \pm 5, \pm 10\}$

	1	2	2	−4	−11	10
		1	3	5	1	−10
1	1	3	5	1	−10	⌊0
		1	4	9	10	
1	1	4	9	10	⌊0	
		−2	−4	−10		
−2	1	2	5	⌊0		

$D = 4 - 4 \cdot 5 = -16 < 0 \implies x_{12} = -1 \pm 2i$,
nad R : $\underline{f(x) = (x-1)^2(x+2)(x^2+2x+5)}$,
nad C : $\underline{f(x) = (x-1)^2(x+2)(x+1-2i)(x+1+2i)}$

3.

$$\left(\begin{array}{ccc|ccc} 1 & -4 & -3 & 1 & 0 & 0 \\ 1 & -5 & -3 & 0 & 1 & 0 \\ -1 & 6 & 4 & 0 & 0 & 1 \end{array}\right)_{R_2 - R_1, R_3 + R_1} \sim \left(\begin{array}{ccc|ccc} 1 & -4 & -3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right)_{R_3 + 2R_2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -4 & -3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)_{R_1 + 3R_3} \sim \left(\begin{array}{ccc|ccc} 1 & -4 & 0 & -2 & 6 & 3 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)_{-1 \cdot R_2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -4 & 0 & -2 & 6 & 3 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)_{R_1 + 4R_2} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \quad A \cdot B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -2 & 4 & 2 \end{pmatrix}}.$$

4. $\underline{\mathrm{st}\, f} = 2 \cdot 2 + 3 = \underline{7}$