Príklady z Matematiky 2

1 Prvý týždeň

V cvičeniach 1 - 19 vypočítajte integrály priamym integrovaním (pomocou integrálov elementárnych funkcií):

1. $\int \left(3x^3 + 2x - 4\right) dx.$ $\left[\frac{3}{4}x^4 + x^2 - 4x + C\right]$
2. $\int \left(\frac{1}{3}x^2 - \frac{1}{5}x\right) dx.$ $\left[\frac{1}{9}x^3 - \frac{1}{10}x^2 + C\right]$
3. $\int \left(\sqrt{x^3} - \frac{1}{\sqrt{x}}\right) dx.$ $\left[\frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C\right]$
4. $\int \frac{\sqrt{x^4+2+x^{-4}}}{x^3} dx.$ $\left[\ln|x| - \frac{1}{4x^4} + C\right]$
5. $\int \frac{x(\sqrt[3]{x} - x\sqrt[3]{x})}{\sqrt[4]{x}} dx.$ $\left[-\frac{12}{37}\sqrt[12]{x^{37}} + \frac{12}{25}\sqrt[12]{x^{25}} + C\right]$
6. $\int \frac{x^3-1}{x-1} dx.$ $\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C\right]$
7. $\int e^x a^x dx.$ $\left[\frac{e^x a^x}{1+\ln a} + C\right]$
8. $\int \left(5\cos x - 2x^5 + \frac{3}{1+x^2}\right) dx.$ $\left[5\sin x - \frac{1}{3}x^6 + 3\,\mathrm{arctg}\, x + C\right]$
9. $\int \left(10^{-x} + \frac{x^2+2}{x^2+1}\right) dx.$ $\left[-\frac{10^{-x}}{\ln 10} + x + \mathrm{arctg}\, x + C\right]$
10. $\int (2\sin x - 3\cos x)\, dx.$ $[-2\cos x - 3\sin x + C]$
11. $\int \frac{1}{\sqrt{3-3x^2}} dx.$ $\left[\frac{1}{\sqrt{3}}\arcsin x + C\right]$
12. $\int \frac{3.2^x - 2.3^x}{2^x} dx.$ $\left[3x - \frac{3^x}{2^{x-1}(\ln 3 - \ln 2)} + C\right]$
13. $\int \frac{1+\cos^2 x}{1+\cos 2x} dx.$ $\left[\frac{1}{2}\,\mathrm{tg}\, x + \frac{1}{2}x + C\right]$
14. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx.$ $[-\mathrm{cotg}\, x - \mathrm{tg}\, x + C.]$
15. $\int \mathrm{tg}^2 x dx.$ $[\mathrm{tg}\, x - x + C]$
16. $\int \mathrm{cotg}^2 x dx.$ $[-\mathrm{cotg}\, x - x + C]$
17. $\int \frac{dx}{\cos 2x + \sin^2 x}.$ $[\mathrm{tg}\, x + C]$
18. $\int \frac{(1+2x^2)}{x^2(1+x^2)} dx.$ $\left[-\frac{1}{x} + \mathrm{arctg}\, x + C\right]$
19. $\int \frac{(1+x)^2}{x(1+x^2)} dx.$ $[\ln|x| + 2\,\mathrm{arctg}\, x + C]$

2 Druhý týždeň

Vypočítajte integrály

1. $\int \frac{-2x+19}{x^2+x-6}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{-2x+19}{x^2+x-6} = \frac{3}{x-2} - \frac{5}{x+3}\\ \text{Výsledok: } \ln\frac{|x-2|^3}{|x+3|^5} + C\end{array}\right]$

2. $\int \frac{2}{9x^2-1}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{2}{(3x+1)(3x-1)} = \frac{1}{3x-1} - \frac{1}{3x+1} = \frac{1}{3}\frac{1}{x-\frac{1}{3}} - \frac{1}{3}\frac{1}{x+\frac{1}{3}}\\ \text{Výsledok: } \frac{1}{3}\ln\left(x-\frac{1}{3}\right) - \frac{1}{3}\ln\left(x+\frac{1}{3}\right) + C\end{array}\right]$

3. $\int \frac{5x^2-7x+10}{x^3-x^2-4x-6}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{5x^2-7x+10}{x^3-x^2-4x-6} = \frac{2}{x-3} + \frac{3x-2}{x^2+2x+2} =\\ = \frac{2}{x-3} + \frac{3}{2}\frac{2x-\frac{4}{3}}{x^2+2x+2} = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2-2-\frac{4}{3}}{x^2+2x+2} =\\ = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2-2-\frac{4}{3}}{x^2+2x+2} = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2}{x^2+2x+2} - 5\frac{1}{(x+1)^2+1}\\ \text{Výsledok: } 2\ln|x-3| + \frac{3}{2}\ln\left(x^2+2x+2\right) - 5\,\mathrm{arctg}\,(x+1) + C\end{array}\right]$

4. $\int \frac{4x^2+x-13}{2x^3+12x^2+11x+5}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{4x^2+x-13}{2x^3+12x^2+11x+5} = \frac{4x^2+x-13}{(x+5)(2x^2+2x+1)} = \frac{2}{x+5} - \frac{3}{2x^2+2x+1} =\\ = \frac{2}{x+5} - \frac{3}{2}\frac{1}{x^2+x+\frac{1}{2}} = \frac{2}{x+5} - \frac{3}{2}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}} = \frac{2}{x+5} - \frac{\frac{3}{2}}{\frac{1}{4}}\frac{1}{\frac{\left(x+\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}+1} =\\ = \frac{2}{x+5} - 6\frac{1}{(2x+1)^2+1}\\ \text{Výsledok: } 2\ln|x+5| - 3\,\mathrm{arctg}\,(2x+1) + C\end{array}\right]$

5. $\int \frac{2x+1}{x^2+2x+5}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{2x+1}{x^2+2x+5} = \frac{2x+2}{x^2+2x+5} - \frac{1}{(x+1)^2+4}\\ \text{Výsledok: } \ln|x^2+2x+5| - \frac{1}{2}\,\mathrm{arctg}\,\frac{x+1}{2} + C\end{array}\right]$

6. $\int \frac{7-x}{x^3-x^2+3x+5}dx$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{7-x}{x^3-x^2+3x+5} = \frac{1}{x+1} + \frac{2-x}{x^2-2x+5}\\ \text{Výsledok:} \ln\frac{|x+1|}{\sqrt{x^2-2x+5}} + \frac{1}{2}\,\mathrm{arctg}\left(\frac{x-1}{2}\right) + C.\end{array}\right]$

7. $\int \frac{1}{x^3+1}dx$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} + \frac{1}{x^2-x+1}\left(\frac{2}{3} - \frac{1}{3}x\right)\\ \text{Výsledok: } \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{\sqrt{3}}{3}(2x-1) + C.\end{array}\right]$

8. $\int \frac{x^5+x^4-7x^3+8x-3}{x^3+x^2-6x}dx.$ $\left[\begin{array}{c}\text{Metóda: integrovanie racionálnych funkcií.}\\ \frac{x^5+x^4-7x^3+8x-3}{x^3+x^2-6x} = x^2 - 1 + \frac{1}{2x} + \frac{1}{2(x-2)}\\ \text{Výsledok: } \frac{x^3}{3} - x + \frac{1}{2}\ln|x(x-2)| + C\end{array}\right]$

9. $\int \frac{2x^3-2x^2+4x-4}{x^4+4}dx.$ $\left[\begin{array}{c}\text{Návod: } x^4+4 = x^4+4x^2+4-4x^2 = (x^2+2)^2-(2x)^2.\\ \text{Metóda:integrovanie racionálej funkcie.}\\ \frac{2x^3-2x^2+4x-4}{x^4+4} = \frac{x-1}{x^2-2x+2} + \frac{x-1}{2x+x^2+2}\\ \text{Výsledok:}\frac{1}{2}\ln\left[\left(x^2-2x+2\right)\left(x^2+2x+2\right)\right] - 2\,\mathrm{arctg}\,(x+1) + C\end{array}\right]$

10. $\int \frac{6x-13}{(4x^2+4x+17)^2}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \text{Výsledok:} -\frac{x+2}{2(4x^2+4x+17)} - \frac{1}{16}\operatorname{arctg}\frac{2x+1}{4} + C \end{array} \right]$

3 Tretí týždeň

Vypočítajte integrály:

1. $\int x \operatorname{arctg} x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = \operatorname{arctg} x, f'(x) = \frac{1}{1+x^2}, g'(x) = x, g(x) = \frac{x^2}{2} \\ \text{Výsledok :}\frac{x^2}{2}\operatorname{arctg} x + \frac{1}{2}\operatorname{arctg} x - \frac{1}{2}x + C \end{array} \right]$

2. $\int xe^{2x}dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = x, f'(x) = 1, g'(x) = e^{2x}\sin x, g(x) = \frac{1}{2}e^{2x} \\ \text{Výsledok : } \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \end{array} \right]$

3. $\int\limits_0^{\frac{\pi}{2}} e^{2x}\sin x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = e^{2x}, f'(x) = 2e^{2x}, g'(x) = \sin x, g(x) = -\cos x \\ \text{Výsledok : } \frac{1}{5}(1+2e^{\pi}) + C \end{array} \right]$

4. $\int \ln x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = \ln x, f'(x) = \frac{1}{x}, g'(x) = 1, g(x) = x \\ \text{Výsledok : } x\ln x - x + C \end{array} \right]$

5. $\int_1^{\sqrt{3}} x \operatorname{arctg} x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = \operatorname{arctg} x, f'(x) = \frac{1}{1+x^2}, g'(x) = x, g(x) = \frac{x^2}{2} \\ \text{Výsledok : } \frac{5\pi}{12} - \frac{1}{2}\left(\sqrt{3}-1\right) \end{array} \right]$

6. $\int \operatorname{arccotg} x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = \operatorname{arccotg} x, f'(x) = -\frac{1}{1+x^2}, g'(x) = 1, g(x) = x \\ \text{Výsledok : } x\operatorname{arccotg} x + \frac{1}{2}\ln\left(x^2+1\right) + C \end{array} \right]$

7. $\int \ln^2 x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes (2x)} \\ f(x) = \ln^2 x, f'(x) = 2\ln x \cdot \frac{1}{x}, g'(x) = 1, g(x) = x \\ \text{Výsledok : } x\ln^2 x - 2x\ln x + 2x + C \end{array} \right]$

8. $\int \frac{x}{\cos^2 x}dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = x, f'(x) = 1, g'(x) = \frac{1}{\cos^2 x}, g(x) = \operatorname{tg} x \\ \text{Výsledok : } x\operatorname{tg} x + \ln|\cos x| + C \end{array} \right]$

9. $\int\limits_0^{\frac{\sqrt{2}}{2}} \frac{\arcsin x}{\sqrt{1+x}}dx.$ $\left[\begin{array}{c} \text{Metóda: per partes} \\ f(x) = \arcsin x, f'(x) = \frac{1}{\sqrt{1-x^2}}, g'(x) = (1+x)^{-\frac{1}{2}}, g(x) = 2\sqrt{1+x} \\ \text{Výsledok : } \frac{\pi}{2}\sqrt{1+\frac{\sqrt{2}}{2}} + 4\sqrt{1-\frac{\sqrt{2}}{2}} - 4 \end{array} \right]$

10. $\int \arccos x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes + substitúcia}\\ f(x)=\arccos x, f'(x)=-\frac{1}{\sqrt{1-x^2}}, g'(x)=1, g(x)=x\\ \text{Výsledok : } x\arccos x-\sqrt{1-x^2}+C\end{array}\right]$

11. $\int \frac{\cos x}{\sin^2 x+5\sin x-6}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\sin x, dt=\cos x dx\\ \text{Výsledok : } \frac{1}{7}\ln\frac{1-\sin x}{\sin x+6}\end{array}\right]$

12. $\int\limits_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x+\sin x-6}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\sin x, dt=\cos x dx, x=0\rightarrow\ t=0, x=\frac{\pi}{2}\rightarrow t=1\\ \text{Výsledok : } \frac{1}{5}\ln\frac{3}{8}\end{array}\right]$

13. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{6-5\cos x+\cos^2 x}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\cos x, dt=-\sin x dx, x=0\rightarrow\ t=1, x=\frac{\pi}{2}\rightarrow t=0\\ \text{Výsledok : } \ln\frac{4}{3}\end{array}\right]$

14. $\int \frac{e^x(e^x-1)}{e^{2x}+1}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=e^x, dt=e^x dx\\ \text{Výsledok : } \frac{1}{2}\ln\left(e^{2x}+1\right)-\operatorname{arctg}(e^x)+C\end{array}\right]$

15. $\int \frac{-2\sin x}{\cos^2 x+2\cos x+5}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\cos x, dt=-\sin x dx\\ \text{Výsledok : } \operatorname{arctg}\left(\frac{\cos x+1}{2}\right)+C\end{array}\right]$

16. $\int \frac{1}{\sin x}dx.$ $\left[\begin{array}{c}\int\frac{1}{\sin x}dx=\int\frac{\sin x}{\sin^2 x}dx=\int\frac{\sin x}{1-\cos^2 x}dx\\ \text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\cos x, dt=-\sin x dx\\ \text{Výsledok : } \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C\end{array}\right]$

4 Štvrtý týždeň

1. $\int \frac{2}{x(\ln x-2)(\ln^2 x-2\ln x+2)}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná+integrovanie racionálnych funkcií}\\ t=\ln x, dt=\frac{1}{x}dx,\\ \text{Výsledok : } \ln\frac{|\ln x-2|}{\left((\ln x-1)^2+1\right)^{\frac{1}{2}}}-\operatorname{arctg}(\ln x-1)+C\end{array}\right]$

2. $\int \frac{1}{\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg} t, dx=\frac{2}{1+t^2}dt, \cos t=\frac{1-t^2}{1+t^2}.\\ \text{Výsledok : } \ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right|+C\end{array}\right]$

3. $\int_1^{64} \frac{2\sqrt{x}}{x\left(\sqrt[3]{x}+\sqrt{x}\right)}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná}\\ t=\sqrt[6]{x}, x=t^6, dx=6t^5dt, x^{\frac{1}{3}}=t^2, x^{\frac{1}{2}}=t^3.\\ \text{Výsledok: } 12\ln\frac{3}{2}\end{array}\right]$

4. $\int \frac{x}{1+\sqrt{x-1}}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná+ integrovanie racionálnych funkcií}\\ t=\sqrt{x-1}, x=t^2+1, dx=2tdt.\\ \text{Výsledok: } \frac{2}{3}(x-1)^{\frac{3}{2}}-x+1+4\sqrt{x-1}-4\ln\left(1+\sqrt{x-1}\right)+C\end{array}\right]$

5. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}.$ $\left[\begin{array}{c} \text{Metóda: substitučná+ integrovanie racionálnych funkcií} \\ t=\sqrt{\frac{1-x}{1+x}}, x=\frac{1-t^2}{1+t^2}, dx=-\frac{4t}{(1+t^2)^2}dt. \\ \text{Výsledok:} \quad \ln\left|\frac{\sqrt{\frac{1-x}{1+x}}-1}{\sqrt{\frac{1-x}{1+x}}+1}\right| + 2\,\text{arctg}\sqrt{\frac{1-x}{1+x}}+C \end{array}\right]$

6. $\int \sqrt{x^2+4x+3}\,dx.$ $\left[\begin{array}{c} \text{Metóda: Eulerova substitúcia} \\ \sqrt{x^2+4x+3}=x-t, x=\frac{t^2-3}{2(t+2)}, dx=\frac{2t^2+8t+6}{4(t+2)^2}dt. \\ \text{Výsledok:} \ -\frac{1}{8}\left(x-\sqrt{x^2+4x+3}\right)^2-\frac{1}{2}\left(x-\sqrt{x^2+4x+3}\right)+ \\ +\frac{1}{2}\ln\left|x-\sqrt{x^2+4x+3}+2\right|+\frac{1}{8\left(x-\sqrt{x^2+4x+3}+2\right)^2}+C \end{array}\right]$

7. $\int \frac{1}{x-\sqrt{x^2-x+1}}\,dx.$ $\left[\begin{array}{c} \text{Metóda: Eulerova substitúcia} \\ \sqrt{x^2-x+1}=x+t, x=\frac{1-t^2}{2t+1}, dx=\frac{-2t^2-2t-2}{(2t+1)^2}dt. \\ \text{Výsledok:} \ 2\ln\left|x-\sqrt{x^2-x+1}\right|-2\ln\left|2\sqrt{x^2-x+1}-2x+1\right|+ \\ +\frac{1}{2\left(\sqrt{x^2-x+1}-2x+1\right)}+C \end{array}\right]$

8. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3+\cos x}dx.$ $\left[\begin{array}{c} \text{Metóda: univerzálna trigonometrická substitúcia+} \\ \text{+integrovanie racionálnej funkcie.} \\ t=\text{tg}\,\frac{x}{2}, x=2\,\text{arctg}\,t, dx=\frac{2}{1+t^2}dt, \cos t=\frac{1-t^2}{1+t^2}. \\ x=\frac{\pi}{3}\rightarrow\ t=\frac{1}{\sqrt{3}}, x=\frac{\pi}{2}\rightarrow\ t=1, \\ \text{Výsledok:}\frac{1}{\sqrt{2}}\left[\frac{\pi}{4}-\text{arctg}\,\frac{1}{\sqrt{6}}\right] \end{array}\right]$

9. $\int \frac{1}{\sin x-\cos x}dx.$ $\left[\begin{array}{c} \text{Metóda: univerzálna trigonometrická substitúcia+} \\ \text{+integrovanie racionálnej funkcie.} \\ t=\text{tg}\,\frac{x}{2}, x=2\,\text{arctg}\,t, dx=\frac{2}{1+t^2}dt, \\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}. \\ \text{Výsledok}:\frac{1}{\sqrt{2}}\ln\left|\frac{\text{tg}\,\frac{x}{2}+1-\sqrt{2}}{\text{tg}\,\frac{x}{2}+1+\sqrt{2}}\right|+C \end{array}\right]$

10. $\int \frac{1-\sin x}{1+\cos x}dx.$ $\left[\begin{array}{c} \text{Metóda: univerzálna trigonometrická substitúcia+} \\ \text{+integrovanie racionálnej funkcie.} \\ t=\text{tg}\,\frac{x}{2}, x=2\,\text{arctg}\,t, dx=\frac{2}{1+t^2}dt, \\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}. \\ \text{Výsledok}: \ \text{tg}\,\frac{x}{2}-\ln\left(\text{tg}^2\,\frac{x}{2}+1\right)+C \end{array}\right]$

11. $\int_0^{\frac{\pi}{2}} \frac{1}{3-2\sin x+\cos x}dx.$ $\left[\begin{array}{c} \text{Metóda: univerzálna trigonometrická substitúcia+} \\ \text{+integrovanie racionálnej funkcie.} \\ t=\text{tg}\,\frac{x}{2}, x=2\,\text{arctg}\,t, dx=\frac{2}{1+t^2}dt, \\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}. \\ x=0\rightarrow\ t=0, x=\frac{\pi}{2}\rightarrow\ t=1, \\ \text{Výsledok}:\frac{\pi}{4} \end{array}\right]$

12. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\operatorname{tg}^2 x}{(1+\operatorname{tg} x)^2} dx.$ $\left[\begin{array}{c} \text{Metóda:trigonometrická substitúcia+integrovanie racionálnej funkcie.} \\ t = \operatorname{tg} x, x = \operatorname{arctg} t, dx = \frac{1}{1+t^2} dt, \\ x = \frac{\pi}{4} \to \ t = 1, x = \frac{\pi}{3} \to \ t = \sqrt{3}. \\ \text{Výsledok :} \frac{2-\sqrt{3}}{2} \end{array}\right]$

13. $\int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3} dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = e^x - 1, dt = e^x dx, x = 0 \to \ t = 0, x = \ln 5 \to t = 4 \\ s = \sqrt{t}, t = s^2, dt = 2sds, t = 0 \to s = 0, t = 4 \to s = 2 \\ \text{Výsledok : } 4 - \pi \end{array}\right]$

14. $\int \frac{e^x+10}{e^{2x}-2e^x+5} dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná+integrovanie racionálnych funkcií} \\ \text{Výsledok : } 2x - \ln\left(e^{2x} - 2e^x + 5\right) + \frac{3}{2} \operatorname{arctg}\left(\frac{1}{2}e^x - \frac{1}{2}\right) + C \end{array}\right]$

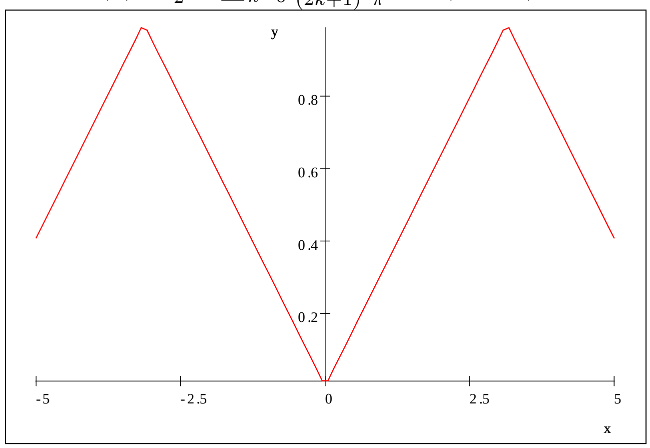
15. $\int x^2 e^{\sqrt{x}} dx.$ $\left[\begin{array}{c} \text{Metóda: substitúcia + per partes} \\ \text{Výsledok : } \left[2\left(\sqrt{x}\right)^5 - 10x^2 + 40\left(\sqrt{x}\right)^3 - 120x + 240\sqrt{x} - 240\right] e^{\sqrt{x}} + C \end{array}\right]$

5 Piaty týždeň

Vo výsledkoch všetkých príkladov nie je načrtnutý graf periodického normalizovaného pokračovania, $\overline{f} : \boldsymbol{R} \to \boldsymbol{R},$ ku ktorému konverguje Fourierov rad, ale pre lepšie pochopenie **len súčet konečného počtu členov** Fourierovho radu. Graf funkcie $\overline{f}$ musí každý študent načrtnúť samostatne.

1. Vypočítajte Fourierov rad funkcie $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $f(x) = |x|$ pre interval $\langle -1, 1\rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

$$a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1}|x|\cos\frac{2n\pi x}{2}dx,\ n = 0,1,2,\dots$$
$$b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1}|x|\sin\frac{2n\pi x}{2}dx,\ n = 1,2,\dots$$
$$a_0 = 1,$$
$$a_n = 2\left(\frac{1}{n\pi}\right)^2\left((-1)^n - 1\right) = \begin{cases} \frac{-4}{n^2\pi^2},\ n = 2k+1, k = 0,1,2,\dots \\ 0,\ n = 2k, k = 1,2,\dots \end{cases}$$
$$b_n = 0,\ n = 1,2,\dots$$
$$\overline{f}(x) = \frac{1}{2} + \sum_{k=0}^{\infty}\frac{-4}{(2k+1)^2\pi^2}\cos(2k+1)\pi x$$



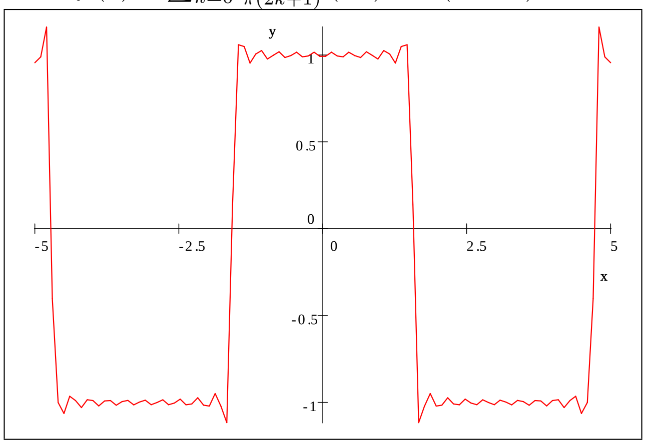
Graf súčtu prvých dvadsiatich členov F.R.

2. Vypočítajte Fourierov rad funkcie

$$f : \boldsymbol{R} \longrightarrow \boldsymbol{R},\ f(x) = \begin{cases} \frac{\cos\pi x}{|\cos\pi x|}, & x \neq \frac{1}{2} + k \\ 0, & x = \frac{1}{2} + k \end{cases}$$

pre interval $\langle -1, 1\rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

$$\left[\begin{array}{c} a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1} f(x)\cos\frac{2n\pi x}{2}dx,\ n = 0,1,2,\dots \\ b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1} f(x)\sin\frac{2n\pi x}{2}dx,\ n = 1,2,\dots \\ a_0 = 0, \\ a_n = \frac{4}{2}\left(\int_0^{\frac{1}{2}}\cos n\pi x dx - \int_{\frac{1}{2}}^{1}\cos n\pi x dx\right) = \frac{4}{\pi n}\left(\sin\frac{\pi n}{2}\right) = \\ = \begin{cases} \frac{4}{\pi(2k+1)}(-1)^k,\ n = 2k+1, k = 0,1,2,\dots \\ 0,\ n = 2k, k = 1,2,\dots \end{cases} \\ b_n = 0,\ n = 1,2,\dots \\ \overline{f}(x) = \sum_{k=0}^{\infty}\frac{4}{\pi(2k+1)}(-1)^k\cos(2k+1)\pi x \end{array}\right]$$



3. Vypočítajte Fourierov rad funkcie $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $f(x) = x$ pre interval $\langle -\pi, \pi \rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

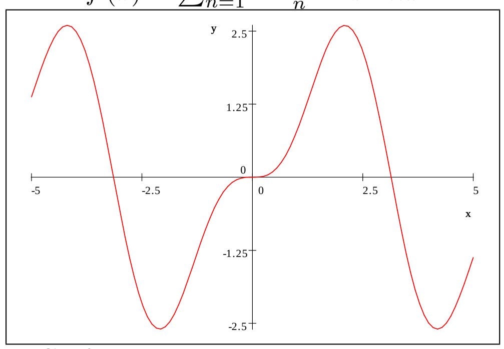
$$a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\cos\frac{2n\pi x}{2\pi}dx,\ n = 0,1,2,\dots$$

$$b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin\frac{2n\pi x}{2\pi}dx,,\ n = 1,2,\dots$$

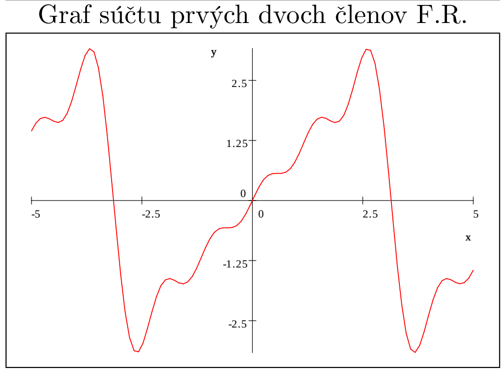
$$a_n = 0,\ n = 0,1,2,\dots$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin nx dx = \frac{2}{n}(-1)^{n+1},\ n = 1,2,\dots$$

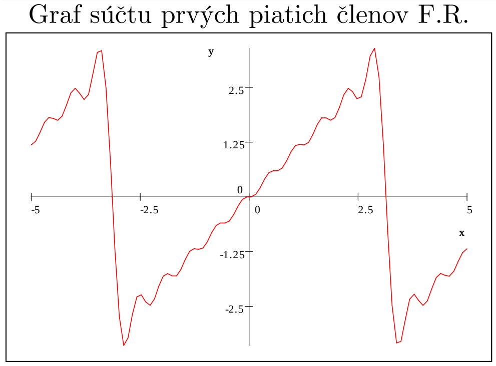
$$\overline{f}(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$



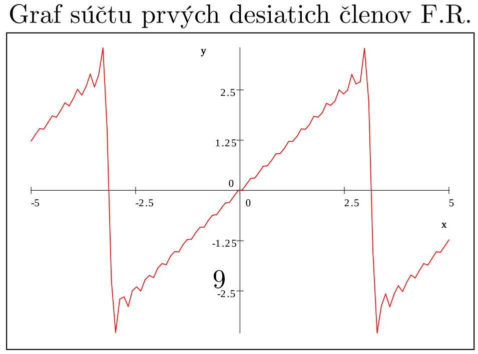
Graf súčtu prvých dvoch členov F.R.



Graf súčtu prvých piatich členov F.R.



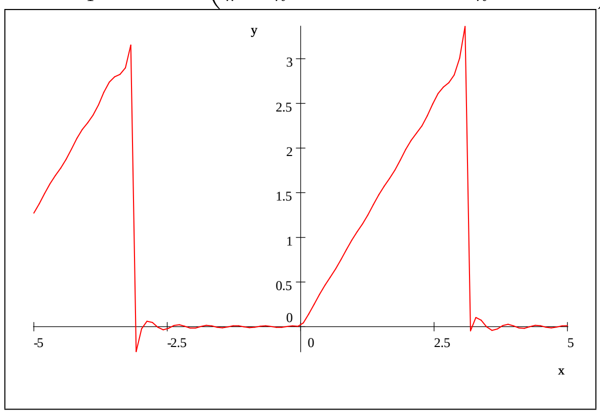
Graf súčtu prvých desiatich členov F.R.



Graf súčtu prvých dvadsiatich členov F.R.

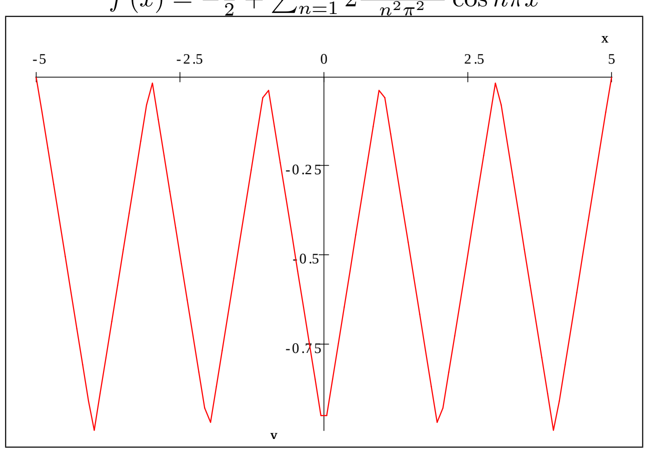
4. Vypočítajte Fourierov rad funkcie $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $f(x) = \frac{x+|x|}{2}$ pre interval $\langle -\pi, \pi \rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

$$a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{2\pi}\int_0^{\pi} x\cos\frac{2n\pi x}{2\pi}dx,\ n = 0, 1, 2, \ldots$$
$$b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{2\pi}\int_0^{\pi} x\sin\frac{2n\pi x}{2\pi}dx,\ n = 1, 2, \ldots$$
$$a_0 = \frac{\pi}{2},$$
$$a_n = \frac{1}{\pi}\frac{(-1)^n - 1}{n^2},\ n = 1, 2, \ldots$$
$$b_n = \frac{(-1)^{n+1}}{n},\ n = 1, 2, \ldots$$
$$\overline{f}(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{(-1)^n - 1}{n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right)$$



5. Vypočítajte Fourierov rad funkcie $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $f(x) = |x| - 1$ pre interval $\langle -1, 1 \rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

$$a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1} (|x| - 1)\cos\frac{2n\pi x}{2}dx,\ n = 0, 1, 2, \ldots$$
$$b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{2}\int_{-1}^{1} (|x| - 1)\sin\frac{2n\pi x}{2}dx,\ n = 1, 2, \ldots$$
$$a_0 = -1,$$
$$a_n = \frac{2(-1)^n - 2}{n^2\pi^2},\ n = 1, 2, \ldots$$
$$b_n = 0,\ n = 1, 2, \ldots$$
$$\overline{f}(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x$$



6. Vypočítajte Fourierov rad funkcie

$$f : \boldsymbol{R} \longrightarrow \boldsymbol{R},\ f(x) = \begin{cases} x, & x \in (-\infty, 1\rangle \\ 1, & x \in (1, 2\rangle \\ 3 - x, & x \in (2, \infty) \end{cases}$$

pre interval $\langle 0, 3\rangle$ a načrtnite graf funkcie, ku ktorej rad konverguje.

$$a_n = \frac{2}{l}\int_a^{a+l} f(x)\cos\frac{2n\pi x}{l}dx = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}dx,\ n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{l}\int_a^{a+l} f(x)\sin\frac{2n\pi x}{l}dx = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}dx,\ n = 1, 2, \ldots$$

$$a_0 = \frac{2}{3}\left(\int_0^1 x dx + \int_1^2 1 dx + \int_2^3 (3-x)\, dx\right) = \frac{4}{3}$$

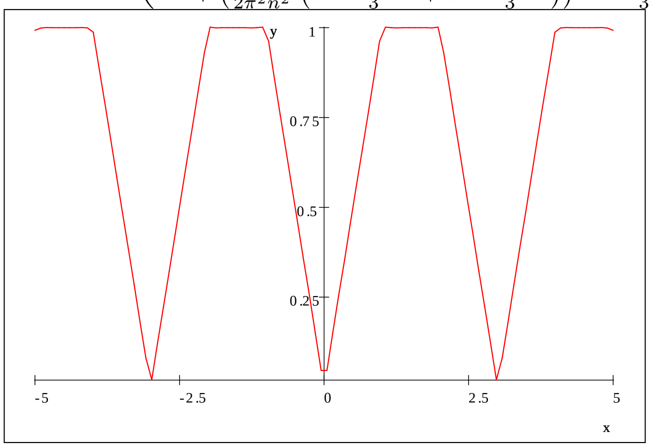
$$a_n = \frac{2}{3}\left(\int_0^1 x\cos\frac{2n\pi x}{3}dx + \int_1^2 1\cos\frac{2n\pi x}{3}dx + \int_2^3 (3-x)\cos\frac{2n\pi x}{3}dx\right)$$

$$= \frac{1}{2}\frac{3\cos\frac{2}{3}n\pi + 2n\pi\sin\frac{2}{3}n\pi - 3}{n^2\pi^2} - \frac{-\sin\frac{4}{3}n\pi + \sin\frac{2}{3}n\pi}{n\pi} - \frac{1}{2}\frac{6\cos^2 n\pi - 3 + 2n\pi\sin\frac{4}{3}n\pi - 3\cos\frac{4}{3}n\pi}{n^2\pi^2} =,\ n = 1, 2, \ldots$$

$$b_n = \frac{2}{3}\left(\int_0^1 x\sin\frac{2n\pi x}{3}dx + \int_1^2 1\sin\frac{2n\pi x}{3}dx + \int_2^3 (3-x)\sin\frac{2n\pi x}{3}dx\right)$$

$$= -\frac{1}{2}\frac{-3\sin\frac{2}{3}n\pi + 2n\pi\cos\frac{2}{3}n\pi}{n^2\pi^2} + \frac{-\cos\frac{4}{3}n\pi + \cos\frac{2}{3}n\pi}{n\pi} + \frac{1}{2}\frac{-6\sin n\pi\cos n\pi + 2n\pi\cos\frac{4}{3}n\pi + 3\sin\frac{4}{3}n\pi}{n^2\pi^2},\ n = 1, 2, \ldots$$

$$\overline{f}(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\left(\begin{array}{c} \left(\frac{1}{2\pi^2 n^2}\left(3\cos\frac{2}{3}\pi n - 6 + 3\cos\frac{4}{3}\pi n\right)\right)\cos\frac{2n\pi x}{3} + \\ + \left(\frac{1}{2\pi^2 n^2}\left(3\sin\frac{2}{3}\pi n + 3\sin\frac{4}{3}\pi n\right)\right)\sin\frac{2n\pi x}{3} \end{array}\right)$$

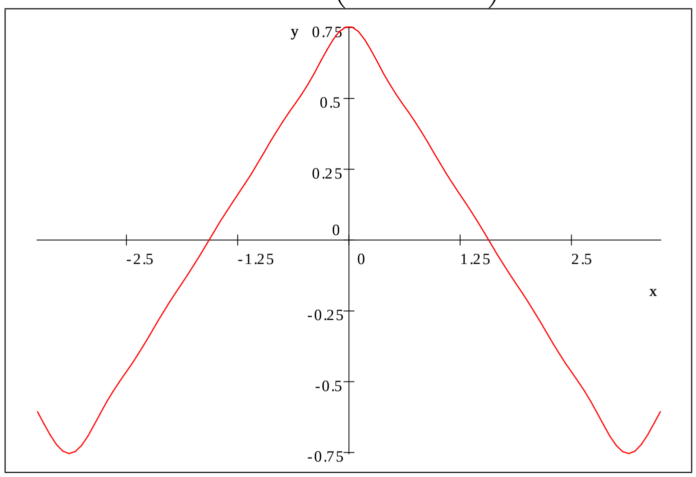


7. Rozložte funkciu $f : \boldsymbol{R} \longrightarrow \boldsymbol{R},\ f(x) = \frac{\pi}{4} - \frac{x}{2}$ na intervale $\langle 0, \pi\rangle$ do

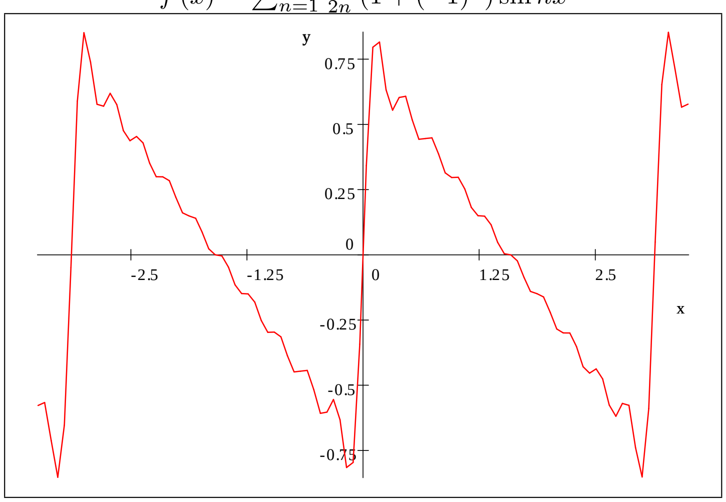
a) kosínusového Fourierovho radu,

b) do sínusového Fourierovho radu a načrtnite grafy funkcií ku ktorým tieto rady konvergujú.

$$a_0 = \frac{4}{2\pi}\int_0^\pi \left(\frac{\pi}{4} - \frac{x}{2}\right) dx = 0$$
$$a_n = \frac{4}{2\pi}\int_0^\pi \left(\frac{\pi}{4} - \frac{x}{2}\right) \cos nx dx = \frac{1}{n^2\pi}\left((-1)^{n+1} + 1\right),\ n = 1, 2, \ldots$$
$$b_n = 0$$
$$\overline{f}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi}\left((-1)^{n+1} + 1\right) \cos nx$$



$$a_n = 0,\ n = 0, 1, 2, \ldots$$
$$b_n = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{4} - \frac{x}{2}\right) \sin nx dx = \frac{1}{2n}\left(1 + (-1)^n\right),\ n = 1, 2, \ldots$$
$$\overline{f}(x) = \sum_{n=1}^{\infty} \frac{1}{2n}\left(1 + (-1)^n\right) \sin nx$$



8. Rozložte funkciu

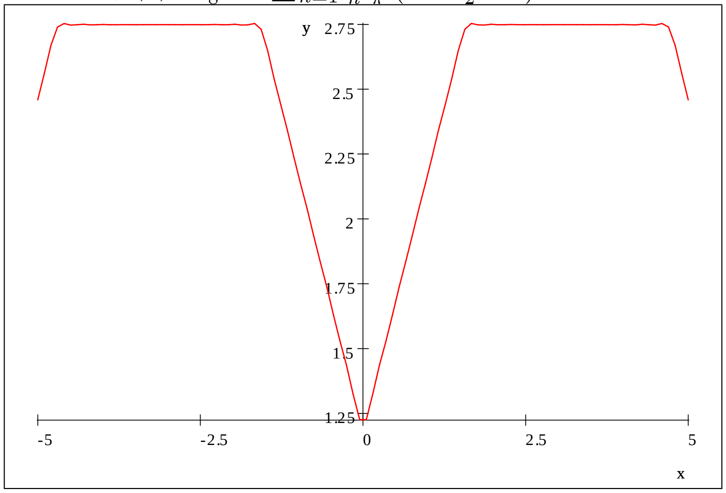
$$f : \boldsymbol{R} \longrightarrow \boldsymbol{R},\ f(x) = \begin{cases} x, & x \in \left(-\infty, \frac{\pi}{2}\right\rangle \\ \frac{\pi}{2}, & x \in \left(\frac{\pi}{2}, \infty\right) \end{cases}$$

na intervale $\langle 0, \pi \rangle$ do

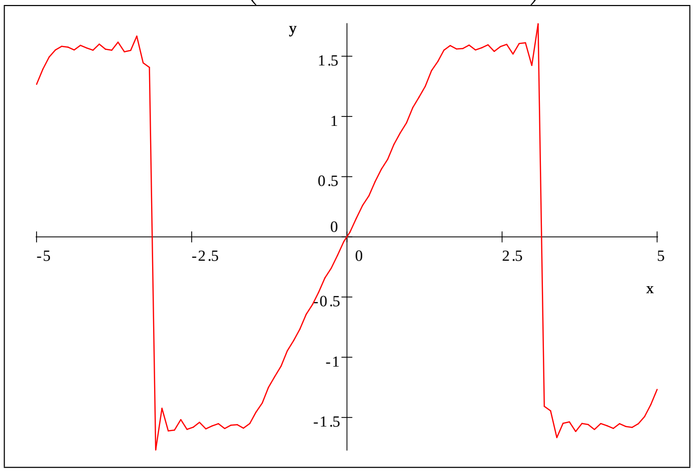
a) kosínusového Fourierovho radu,

b) do sínusového Fourierovho radu a načrtnite grafy funkcií ku ktorým tieto rady konvergujú.

$$a_0 = \frac{4}{2\pi}\left(\int_0^{\frac{\pi}{2}} xdx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}dx\right) = \frac{3}{4}\pi$$
$$a_n = \frac{4}{2\pi}\int_0^{\pi}\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos nxdx = \frac{2}{n^2\pi}\left(\cos n\frac{\pi}{2} - 1\right),\ n = 1, 2, \ldots$$
$$b_n = 0$$
$$\overline{f}(x) = \frac{3}{8}\pi + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left(\cos n\frac{\pi}{2} - 1\right)\cos nx$$



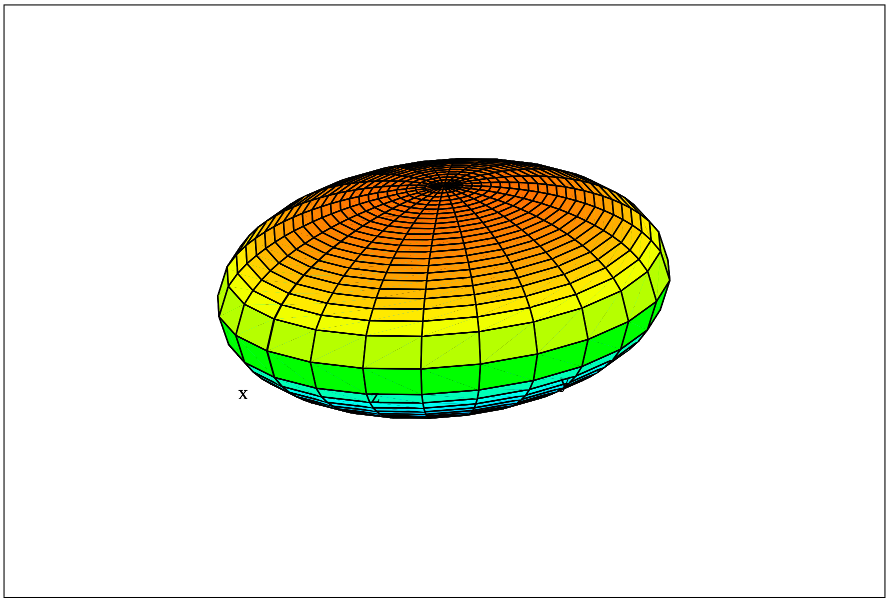
$$a_n = 0,\ n = 0, 1, 2, \ldots$$
$$b_n = \frac{4}{2\pi}\left(\int_0^{\frac{\pi}{2}} x\sin nxdx + \int_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\sin nxdx\right)$$
$$= \frac{1}{n}(-1)^{n+1} + \frac{2}{\pi n^2}\sin n\frac{\pi}{2},\ n = 1, 2, \ldots$$
$$\overline{f}(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n}(-1)^{n+1} + \frac{2}{\pi n^2}\sin n\frac{\pi}{2}\right)\sin nx$$



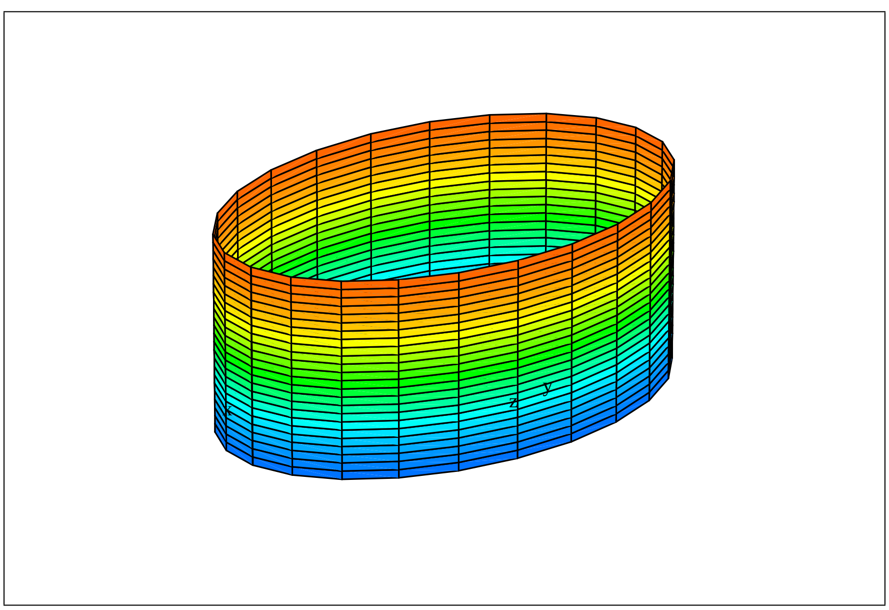
6 Šiesty týždeň

Kvadratické plochy:

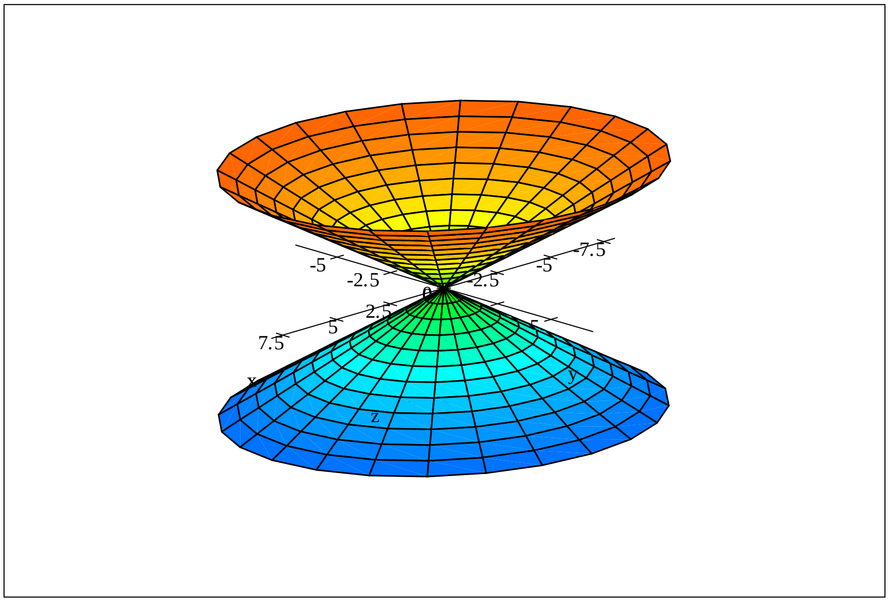
Elipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $\left(\frac{x^2}{9} + \frac{y^2}{6} + \frac{z^2}{4} = 1\right)$



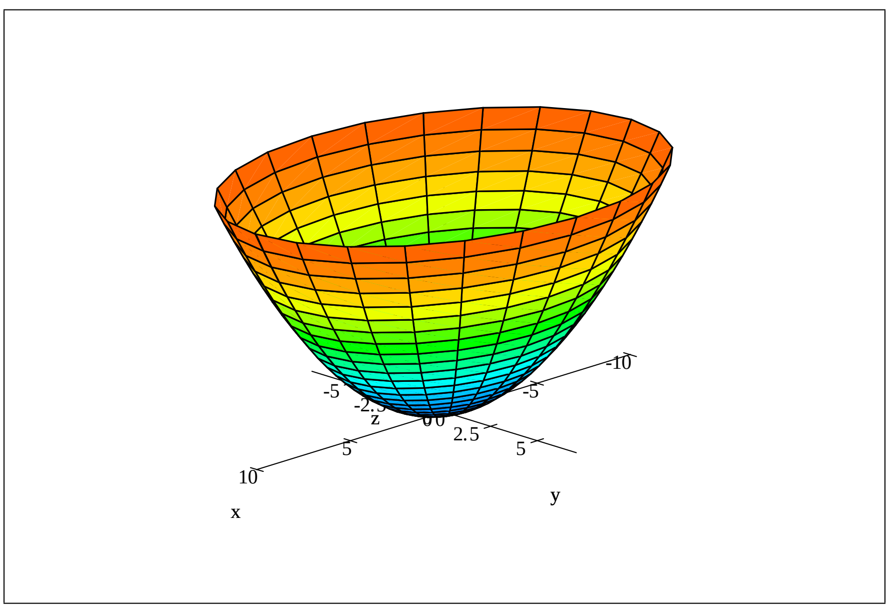
Eliptická valcová plocha $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\left(\frac{x^2}{4} + \frac{y^2}{3} = 1\right)$



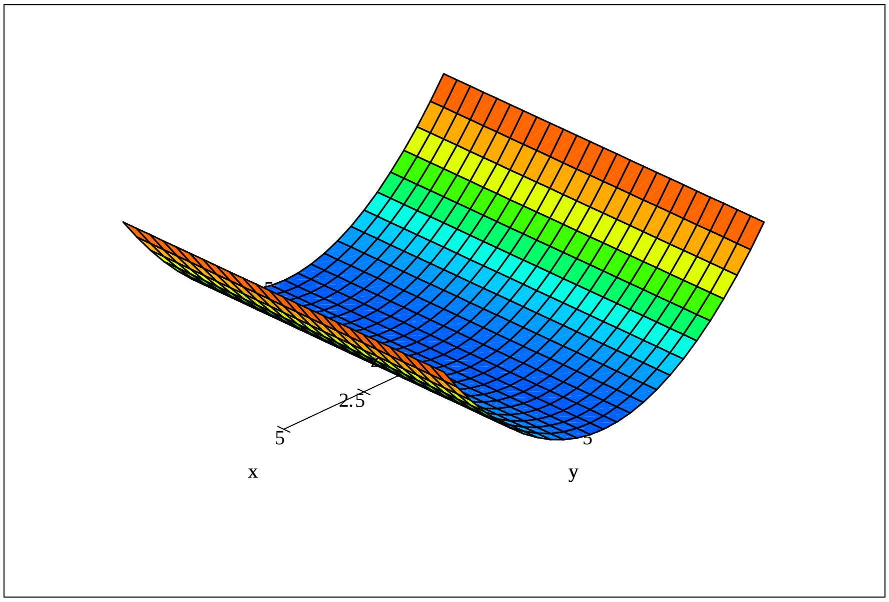
Eliptická kužeľová plocha $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z^2}{4} = 0\right)$



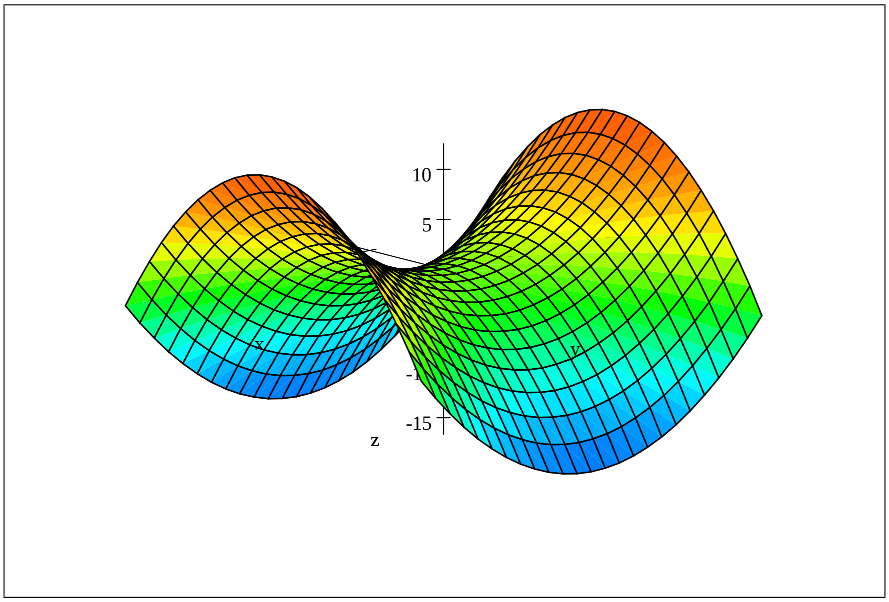
Eliptický paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z}{1} = 0\right)$



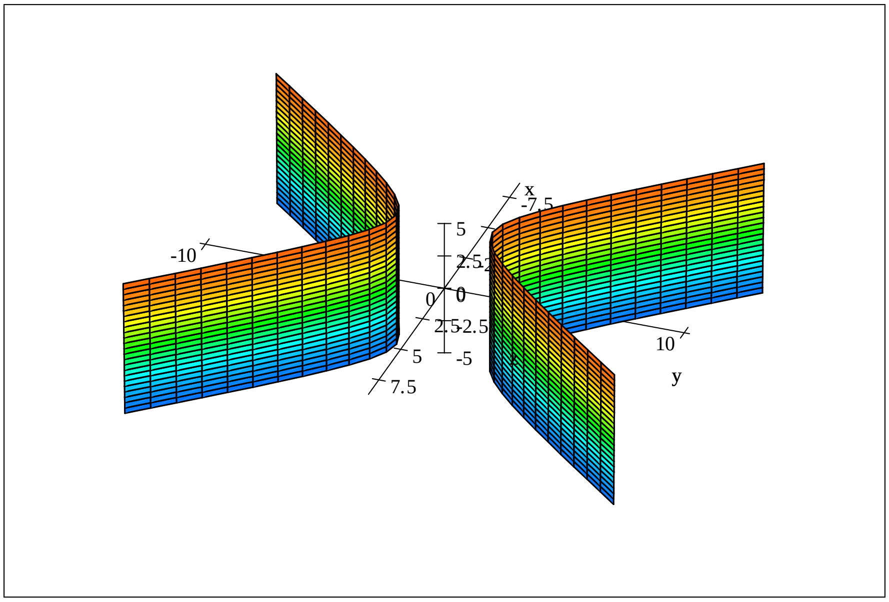
Parabolická valcová plocha $z = ax^2$, $\left(z = \frac{1}{4}x^2\right)$



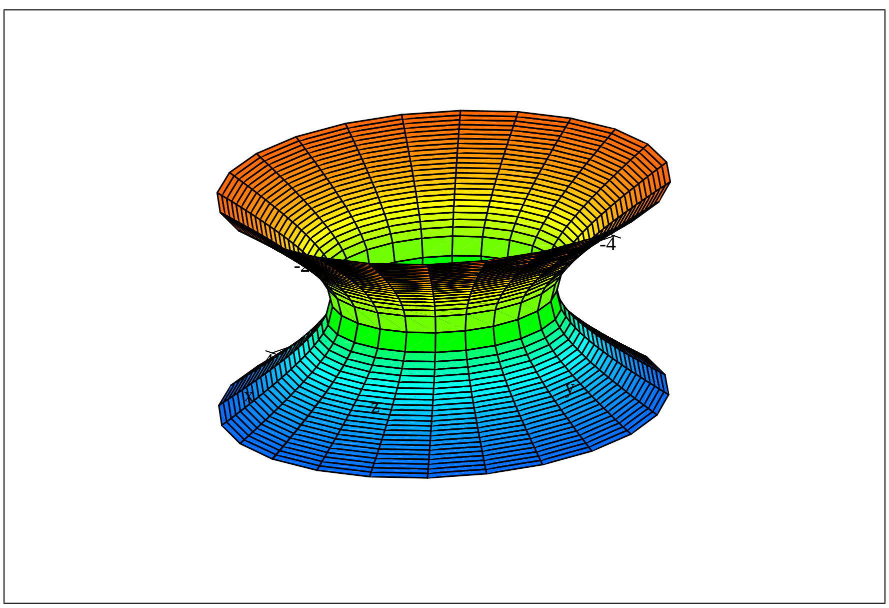
Hyperbolický paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$, $\left(\frac{x^2}{4} - \frac{y^2}{3} - \frac{z}{2} = 0\right)$



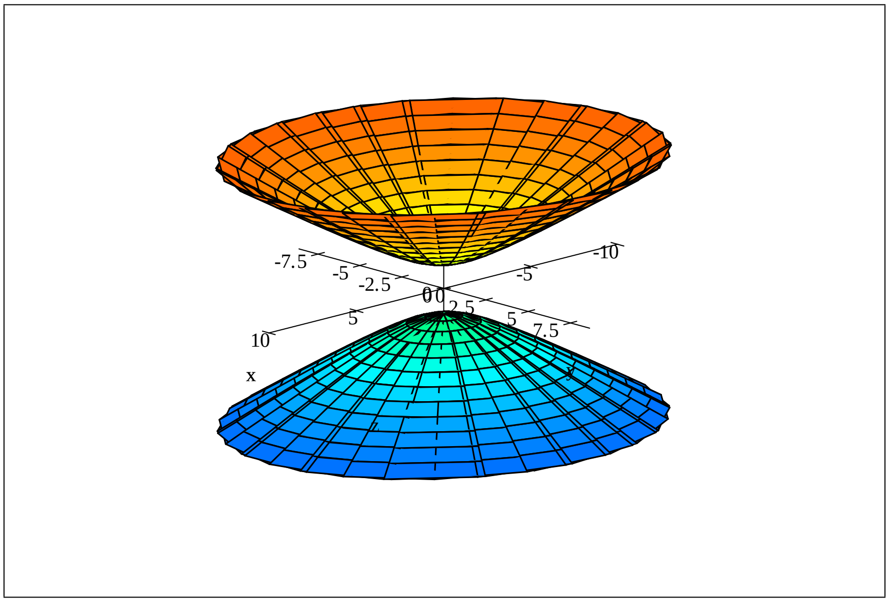
Hyperbolická valcová plocha $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $\left(\frac{y^2}{4} - \frac{x^2}{3} = 1\right)$



Jednodielny hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z^2}{5} = 1\right)$



Dvojdielny hyperboloid $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\left(\frac{z^2}{5} - \frac{x^2}{4} - \frac{y^2}{3} = 1\right)$



Nájdite definičný obor daných funkcií (aj načrtnite):

1. $f(x,y) = \sqrt{x^2 + y^2 - r^2}$, kde $r \geq 0$ je reálne číslo. $\left[D(f) = \left\{(x,y) \in \mathbf{R}^2;\ x^2 + y^2 \geq r^2\right\}\right]$
2. $g(x,y) = \frac{1}{\sqrt{r^2 - x^2 - y^2}}$, kde $r \geq 0$ je reálne číslo. $\left[D(g) = \left\{(x,y) \in \mathbf{R}^2;\ x^2 + y^2 < r^2\right\}\right]$
3. $f(x,y) = \ln(-x-y)$. $\left[D(f) = \left\{(x,y) \in \mathbf{R}^2;\ x < -y\right\}\right]$
4. $f(x,y,z) = \ln\left(1 - x^2 - y^2 + z^2\right)$. $\left[D(f) = \left\{(x,y,z) \in \mathbf{R}^3;\ x^2 + y^2 - z^2 < 1\right\}\right]$
5. $f(x,y,z) = \arccos(2x-1) + \sqrt{1-y^2} + \sqrt{y} + \ln\left(4 - z^2\right)$.
$\left[D(f) = \left\{(x,y,z) \in \mathbf{R}^3;\ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ -2 < z < 2\right\}\right]$
6. $f(x,y) = \sqrt{4 - x^2 - y^2 + 2x - 4y}$. $\left[D(f) = \left\{(x,y) \in \mathbf{R}^2;\ (x-1)^2 + (y+2)^2 \leq 9\right\}\right]$

Vypočítajte limity:

7. $\lim_{(x,y)\longrightarrow(0,2)} \frac{3y^2 - 3xy - 6y}{1 - \sqrt{x-y+3}}$. $[12]$
8. $\lim_{(x,y)\longrightarrow(3,4)} \frac{4 - \sqrt{x+3y+1}}{15 - x - 3y}$. $\left[\frac{1}{8}\right]$
9. $\lim_{(x,y)\longrightarrow(-2,1)} \frac{(2x+y)^2 - 9}{4xy + 2y^2 + 6y}$. $[-3]$
10. $\lim_{(x,y)\longrightarrow(4,0)} \frac{\operatorname{tg}(xy)}{y}$. $[4]$
11. $\lim_{(x,y)\longrightarrow(0,0)} \frac{2 - \sqrt{4-xy}}{xy}$. $\left[\frac{1}{4}\right]$

12. $\lim_{(x,y,z)\longrightarrow(1,1,1)} \frac{\sin(x+y-z-1)}{x+y-z-1}$. [1]

13. $\lim_{(x,y)\longrightarrow(2,3)} \frac{\sin x+2}{(x^2-y^2+5)^2}$. $[+\infty]$

14. $\lim_{(x,y)\longrightarrow(0,0)} \frac{2xy}{xy+2x-y}$. [neexistuje]

15. $\lim_{(x,y)\longrightarrow(0,0)} \frac{xy^2}{x^2+y^4}$. [neexistuje]

16. Dodefinujte funkciu $f(x,y) = \frac{xy}{3-\sqrt{xy+9}}$ tak, aby bola v bode $(0,0)$ spojitá. $[f(0,0) = -6]$

17. Dodefinujte funkciu $f(x,y) = \frac{x^3-y^3}{x^4-y^4}$ tak, aby bola v bode $(2,2)$ spojitá. $\left[f(2,2) = \frac{3}{8}\right]$

18. Dodefinujte funkciu $f(x,y) = \frac{x^2y^2}{x^2y^2+x-y}$ tak, aby bola v bode $(0,0)$ spojitá. [Funkcia sa nedá dodefinovať v bode $(0,0)$ aby bola spojitá]

19. Daná je funkcia $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ predpisom

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & (x,y) \neq (0,0) \\ 2 & (x,y) = (0,0) \end{cases}.$$

Zistite, či je v bode $(0,0)$ spojitá. [Je spojitá]

20. Daná je funkcia $f : A = \left\{\mathbf{x} \in \mathbf{R}^2 : y \neq 0\right\} \longrightarrow \mathbf{R}$, $f(x,y) = \frac{\sin(6xy)}{y}$. Dodefinujte funkciu v bode $(3,0)$ tak, aby bola v tomto bode spojitá. spojitá. $[f(3,0) = 18]$

21. Daná je funkcia $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ predpisom

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Zistite, či je f spojitá. [Je spojitá všade s výnimkou bodu $(0,0)$]

7 Siedmy týždeň

1. Zistite, či je funkcia $f : \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = \begin{cases} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

   diferencovateľná v bode $(0,0)$.

   [nie je]

2. Zistite, či je funkcia $f : \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = \sqrt{|xy|}$ diferencovateľná v bode $(0,0)$.

   [nie je]

3. Daná je funkcia $f : \mathbf{R}^3 \longrightarrow \mathbf{R},\ f(x,y,z) = \begin{cases} \frac{2x-3y+z^2}{\sqrt{x^2+y^2+z^2}} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$ .Vypočítajte

   (a) parciálne derivácie v bode $(0,0,0)$,

   (b) zistite, či je funkcia v bode $(0,0,0)$ diferencovateľná. $\left[\frac{\partial f}{\partial x}(0,0,0) = \infty,\ \frac{\partial f}{\partial y}(0,0,0) = -\infty,\ \frac{\partial f}{\partial z}(0,0,0)\right.$

4. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = \left(x^2+y\right)\sin(x+y)$ v bode $\mathbf{a} = (0,\pi)$.

   $\left[\frac{\partial f}{\partial x}(0,\pi) = -\pi,\ \frac{\partial f}{\partial y}(0,\pi) = -\pi\right]$

5. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = 4x^3 - 2y^2 + 3xy^2 + 5y$ v bode $\mathbf{a} = (1,2)$.

   $\left[\frac{\partial f}{\partial x}(1,2) = 24,\ \frac{\partial f}{\partial y}(1,2) = 9\right]$

6. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = \begin{cases} \left(x^2+y^2\right)\cos\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (\\ 0 & (x,y) = (\end{cases}$

   v bode $\mathbf{a} = (0,0)$.

   $\left[\frac{\partial f}{\partial x}(0,0) = 0,\ \frac{\partial f}{\partial y}(0,0) = 0\right]$

7. Vypočítajte $\frac{\partial f(x,y)}{\partial x}$ a $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ keď $f(x,y) = \begin{cases} \frac{2x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   $\left[\frac{\partial f(x,y)}{\partial x} = \frac{6x^3+6xy^2-4x^4}{(x^2+y^2)^2},\ \frac{\partial f(0,0)}{\partial x} = 2,\ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{-12x^3y-12xy^3+16x^4y}{(x^2+y^2)^3},\ \frac{\partial^2 f(0,0)}{\partial y \partial x} - \text{neexistuje}\right]$

8. Nech $f : \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = \begin{cases} \left(x^2+y^2\right)\sin\frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   Vypočítajte $\frac{\partial^2 f(0,0)}{\partial y \partial x},\ \frac{\partial^2 f(0,0)}{\partial x \partial y}$.

   $\left[\frac{\partial^2 f(0,0)}{\partial y \partial x} = 0,\ \frac{\partial^2 f(0,0)}{\partial x \partial y} = 0\right]$

9. Vypočítajte rovnicu dotykovej roviny a normály ku grafu funkcie $f(x,y) = xy$ v bode $T = (?, 2, 2)$.

   $[\tau : 2x + y - z - 2 = 0, n : x = 1 + 2t, y = 2 + t, z = 2 - t.]$

10. Vypočítajte rovnicu dotykovej roviny a normály ku grafu funkcie $f(x,y) = 2x^2 + y^2$ v bode $T = (1, 1, ?)$.

    $[\tau : 4x + 2y - z - 3 = 0, n : x = 1 + 4t,\ y = 1 + 2t,\ z = 3 - t]$

11. Vypočítajte maticu diferenciálu funkcie $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ ak $\mathbf{g}(u,v) = \left(uv, \frac{u}{v}, 1\right)$ a $\mathbf{f}(x,y) = (\ln x, \cos y)$.

$$\left[\begin{array}{c} \mathcal{D}\mathbf{h}(x,y) = \mathcal{D}\mathbf{g}(f(x,y)) \cdot \mathcal{D}\mathbf{f}(x,y) = \\ = \begin{pmatrix} \cos y & \ln x \\ \frac{1}{\cos y} & -\frac{\ln x}{\cos^2 y} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & -\sin y \end{pmatrix} = \begin{pmatrix} \frac{\cos y}{x} & -\ln x \sin y \\ \frac{1}{x \cos y} & \frac{\ln x \sin y}{\cos^2 y} \\ 0 & 0 \end{pmatrix} \end{array}\right]$$

8 Ôsmy týždeň

1. Vypočítajte parciálne derivácie, gradient a diferenciál funkcie $f(x,y) = \frac{2}{(3x^2+4y^2)^2}$ v bode $\mathbf{a} = (-1, 1)$. $\left[\begin{array}{c} \frac{\partial f}{\partial x}(-1,1) = \frac{24}{343},\ \frac{\partial f}{\partial y}(-1,1) = -\frac{32}{343},\ grad\ f(-1,1) = \left(\frac{24}{343}, -\frac{32}{343}\right), \\ \mathcal{D}f(-1,1)(\mathbf{h}) = \frac{24}{343}h_1 - \frac{32}{343}h_2, \text{ kde } \mathbf{h} = (h_1, h_2) \end{array}\right]$

2. Vypočítajte $\frac{\partial f(x,y)}{\partial x}$ a $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ keď $f(x,y) = \begin{cases} \frac{2x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

$$\left[\begin{array}{c} \frac{\partial f(x,y)}{\partial x} = \frac{6x^3+6xy^2-4x^4}{(x^2+y^2)^2},\ \frac{\partial f(0,0)}{\partial x} = 2, \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{-12x^3y-12xy^3+16x^4y}{(x^2+y^2)^3},\ \frac{\partial^2 f(0,0)}{\partial y \partial x} - \text{neexistuje} \end{array}\right]$$

3. Nech $f : \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = \begin{cases} \left(x^2+y^2\right)\sin\frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   Vypočítajte $\frac{\partial^2 f(0,0)}{\partial y \partial x}$, $\frac{\partial^2 f(0,0)}{\partial x \partial y}$.

   $\left[\frac{\partial^2 f(0,0)}{\partial y \partial x} = 0,\ \frac{\partial^2 f(0,0)}{\partial x \partial y} = 0\right]$

4. Vypočítajte deriváciu funkcie $f(x,y) = e^y \cos(x+y)$ v bode $\mathbf{a} = \left(\frac{\pi}{2}, 0\right)$ v smere vektora $\mathbf{e} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

   $\left[\frac{df}{d\mathbf{e}}(\mathbf{a}) = f_{.\mathbf{e}}(\mathbf{a}) = -\frac{1}{2}\left(1+\sqrt{3}\right)\right]$

5. Nájdite deriváciu funkcie $f(x,y) = 3x^2 - 6xy + y^2$ v bode $\mathbf{a} = \left(-\frac{1}{3}, -\frac{1}{2}\right)$ v smere ľubovoľného jednotkového vektora $\mathbf{e}$. Zistite v akom smere je derivácia

(a) nulová, $\left[\mathbf{e}_1 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \mathbf{e}_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right]$

(b) najväčšia, $\left[\mathbf{e}_3 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right]$

(c) najmenšia. $\left[\mathbf{e}_4 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right]$

9 Deviaty týždeň

V nasledujúcich príkladoch nájdite lokálne extrémy funkcií

1. $f(x,y) = 2x^3 - xy^2 + 5x^2 + y^2$. $\left[f(0,0) = 0 \text{ relatívne minimum, sedlové body } (1,4), (1,-4), \left(-\frac{5}{3}, 0\right)\right]$

2. $f(x,y) = e^{2x}\left(x + y^2 + 2y\right)$.
   $\left[f\left(\frac{1}{2}, -1\right) = -\frac{e}{2} \text{ relatívne minimum}\right]$

3. $f(x,y) = x^3 + y^3 + 3xy + 2$.
   $[f(-1,-1) = 3 \text{ relatívne maximum, sedlový bod } (0,0)]$

4. $f(x,y) = 5xy + \frac{25}{x} + \frac{8}{y}$, $x > 0$, $y > 0$.
   $\left[f\left(\frac{5}{2}, \frac{4}{5}\right) = 30 \text{ relatívne minimum}\right]$

5. $f(x,y) = \frac{1}{2}y + (47 - x - y)\left(\frac{x}{3} + \frac{y}{4}\right)$.
   $[f(21,20) = 282 \text{ relatívne maximum}]$

6. $f(x,y) = xy(2 - x - y)$.
   $\left[f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{27} \text{ relatívne maximum , sedlové body } (0,0), (0,2), (2,0)\right]$

7. $f(x,y) = e^{-x^2-y^2}\left(2y^2 + x^2\right)$. $\left[f(0,0) = 0 \text{ relatívne minimum, } f(0,1) = \frac{2}{e}, f(0,-1) = \frac{2}{e} \text{ relatívne ma}\right.$

8. $f(x,y) = x^2y^2(3 - 4x + 6y)$.
   $$\left[\begin{array}{c} f\left(\frac{3}{10}, -\frac{1}{5}\right) = \frac{27}{12400} \text{ relatívne maximum,} \\ \text{v bodoch } \left\{(x,0) : x > \frac{3}{4}\right\} \wedge \left\{(0,y) : y < \frac{1}{2}\right\} \text{ sú lokálne maximá, pre ktoré } f(.,.) = 0, \\ \text{v bodoch } \left\{(x,0) : x < \frac{3}{4}\right\} \wedge \left\{(0,y) : y > \frac{1}{2}\right\} \text{ sú lokálne minimá, pre ktoré } f(.,.) = 0, \\ \text{sedlové body } \left(\frac{3}{4}, 0\right) a \left(0, \frac{1}{2}\right). \end{array}\right]$$

9. $f(x,y,z) = x^2 + y^2 + z^2 - xy + 2z + x$.
   $\left[f\left(-\frac{2}{3}, -\frac{1}{3}, -1\right) = -\frac{4}{3} \text{ relatívne minimum}\right]$

10. $f(x,y,z) = 3x^2 + 3x + 2y^2 + 2yz + 2y + 2z^2 - 2z$.
    $\left[f\left(-\frac{1}{2}, -1, 1\right) = -\frac{11}{4} \text{ relatívne minimum}\right]$

11. $f(x,y,z) = 2x^2 + y^2 + 2z - xy - xz$.
    $[(2,1,7) \text{ - sedlový bod}]$

12. $f(x,y,z) = x^2 + y^2 + z^2 + 2x + 4y - 6z$.
    $[f(-1,-2,3) = -14 \text{ relatívne minimum}]$

13. $f(x,y,z) = y^2 + 2z^2 + 2x - xy - xz$.
    $\left[\left(\frac{8}{3}, \frac{4}{3}, \frac{2}{3}\right) \text{ - sedlový bod}\right]$

14. $f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$. $[f(24,-144,-1) = -6913 \text{ relatívne minimum, } (0,0,-1) \text{ - sedlový bo}$