14:00

1. Napíšte množinu všetkých riešení sústavy, ktorej rozšírená matica je

a) [5 bodov] v $\mathbb{R}$: $\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right)$ $\quad x_2 = a,\ x_5 = b,$

$x_4 + b = -1 \implies x_4 = -1 - b$

$x_3 - b = 0 \implies x_3 = b, \quad x_1 - a + 2b - 1 - b - b = 1 \implies x_1 = 2 + a,$

$\boxed{P = \{(2 + a\,,\ a\,,\ b\,, -1 - b\,,\ b) \colon a, b \in \mathbb{R}\}}$

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b) [3] v $\mathbb{C}$: $\left(\begin{array}{cc|c} 1+i & 1 & 1 \\ 0 & i & 1+i \end{array}\right)$

$ix_2 = 1 + i \implies x_2 = \frac{1+i}{i} \cdot \frac{-i}{-i} = 1 - i, \quad (1+i)x_1 + x_2 = (1+i)x_1 + 1 - i = 1 \implies (1+i)x_1 = i \implies x_1 = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{1}{2} + \frac{1}{2}i$ $\boxed{P = \{(\frac{1+i}{2}\,,\ 1 - i)\}}$

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2. [5] Riešte sústavu (v R)

$$\begin{aligned} 3x - 7y - 3z &= 1 \\ -2x + 5y + z &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & -7 & -3 & 1 \\ -2 & 5 & 1 & -1 \end{array}\right) \sim_{R1+R2} \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ -2 & 5 & 1 & -1 \end{array}\right) \sim_{R2+2R1} \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & -3 & -1 \end{array}\right) \sim_{R1+2R2} \left(\begin{array}{ccc|c} 1 & 0 & -8 & -2 \\ 0 & 1 & -3 & -1 \end{array}\right)$$

$\boxed{P = \{(8a - 2, 3a - 1, a) \colon a \in R\}}$

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3. [6] $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix}$. Nájdite A^{-1} a napíšte hodnosť matice A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \sim_{R3-3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -8 & -3 & 0 & 1 \end{array}\right) \sim_{\substack{R1-2R_2 \\ R3+3R_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -2 & -3 & 3 & 1 \end{array}\right) \sim_{\substack{R2+R3 \\ R3/-2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & 3/2 & -3/2 & -1/2 \end{array}\right) \sim_{R1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & -7/2 & -1/2 \\ 0 & 1 & 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & 3/2 & -3/2 & -1/2 \end{array}\right) \implies A^{-1} = -\frac{1}{2}\begin{pmatrix} -5 & 7 & 1 \\ 6 & -8 & -2 \\ -3 & 3 & 1 \end{pmatrix}$$

$\exists A^{-1} \implies \text{rank}(A) = 3$

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4. [4] Vypočítajte determinant $\begin{vmatrix} 1 & 2 & 0 & 0 \\ -3 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{vmatrix} = d$, napr. rozvoj podľa A_{4*}:

$$d = (-1)^5 \begin{vmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{vmatrix} + (-1)^8 \begin{vmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = -16 + 7 = \boxed{-9}$$

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5. [3] Nájdite $a \in \mathbb{R}$, pre ktoré je $c = 1$ koreňom polynómu $f(x) = 2x^6 - ax^4 - x^3 + ax^2 + 3a$.

	2	0	$-a$	-1	a	0	$3a$	koeficienty polynómu f
1		2	2	$2-a$	$1-a$	1	1	
	2	2	$2-a$	$1-a$	1	1	$1+3a=0$	$\implies$ $\boxed{a = -1/3}$

Bez použitia Hornerovej schémy: $f(1) = 2 - a - 1 + a + 3a = 1 + 3a = 0 \implies a = -1/3$

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6. [4] Určte násobnosť koreňa $c = i$ polynómu $f(x) = x^5 - 3x^4 + 2x^3 - 6x^2 + x - 3$.

	1	-3	2	-6	1	-3
i		i	$-1-3i$	$3+i$	$-1-3i$	3
	1	$i-3$	$1-3i$	$-3+i$	$-3i$	0
i		i	$-2-3i$	$6-i$	$3i$	
	1	$2i-3$	$-1-6i$	3	0	
i		i	$-3-3i$	$9-4i$	$3i$	
	1	$3i-3$	$-4-9i$	$12-4i \neq 0$	$\implies$ $\boxed{\text{nás.} = 2}$	

15:00

1. [5 bodov] Napíšte množinu všetkých riešení sústavy, ktorej rozšírená matica je

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right) \qquad x_2 = a,\ x_5 = b,$$

$x_4 + b = 1 \implies x_4 = 1 - b$

$x_3 - b = 0 \implies x_3 = b, \quad x_1 - a + 2b + 1 - b - b = -1 \implies x_1 = a - 2,$

$\boxed{P = \{(a-2\,,\,a\,,\,b\,,1-b\,,\,b) \colon a, b \in \mathbb{R}\}}$

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2. [5] Riešte sústavu (v R)

$$\begin{aligned} -2x - 7y - 3z &= 1 \\ 3x + 5y + z &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} -2 & -7 & -3 & 1 \\ 3 & 5 & 1 & -1 \end{array}\right) \sim_{R1+R2} \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 3 & 5 & 1 & -1 \end{array}\right) \sim_{R2-3R1} \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 11 & 7 & -1 \end{array}\right) \quad \begin{array}{c} x_3 = a \\ 11x_2 + 7a = -1 \end{array} \implies x_2 = -\frac{1}{11} - \frac{7}{11}a$$

$$x_1 = 2x_2 + 2x_3 = -\tfrac{2}{11} - \tfrac{14}{11}a + 2a = -\tfrac{2}{11} + \tfrac{8}{11}a \quad \boxed{P = \left\{\left(-\frac{2}{11} + \frac{8}{11}a, -\frac{1}{11} - \frac{7}{11}a, a\right) \colon a \in R\right\}}$$

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3. [8] $A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & -2 \\ 2 & -6 \end{pmatrix}$.

Vypočítajte AB, BA, $A+B$, D^{-1}, $\det(BA)$.

$$AB = \begin{pmatrix} 1 & -3 \\ -2 & 8 \end{pmatrix}, \qquad BA = \begin{pmatrix} 6 & -2 & 2 \\ -4 & 3 & 0 \\ 4 & -3 & 0 \end{pmatrix}, \quad A + B \not\exists \text{(nie sú rovnakého typu)},$$

najrýchlejšie pomocou determinantov: $D^{-1} = \dfrac{1}{10}\begin{pmatrix} -6 & 2 \\ -2 & -1 \end{pmatrix}$, $\qquad \det(BA) = 0$

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4. [5] Pomocou Cramerovho pravidla riešte sústavu

$$\begin{aligned} x - (1+i)y &= 0 \\ -x + 2i\,y &= 1 \end{aligned}$$

$$d = \begin{vmatrix} 1 & -(1+i) \\ -1 & 2i \end{vmatrix} = 2i - 1 - i = i - 1,$$

$$d_1 = \begin{vmatrix} 0 & -(1+i) \\ 1 & 2i \end{vmatrix} = 1 + i,, \qquad d_2 = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$x = \frac{1+i}{-1+i} = \frac{(1+i)(-1-i)}{(-1+i)(-1-i)} = \frac{-2i}{2} = -i, \qquad y = \frac{1}{-1+i} = -\frac{1}{2} - \frac{1}{2}i$

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5. [3] Nájdite $a \in \mathbb{R}$, pre ktoré je $c = -1$ koreňom polynómu $f(x) = 2x^6 - ax^4 - x^3 + ax^2 + 3a$.

	2	0	$-a$	-1	a	0	$3a$	koeficienty polynómu f
-1		-2	2	$-2+a$	$3-a$	-3	3	
	2	-2	$2-a$	$-3+a$	3	-3	$3+3a=0$	$\implies \boxed{a=-1}$

Alebo $f(-1) = 2 - a + 1 + a + 3a = 3 + 3a = 0 \implies a = -1$

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6. [4] Určte násobnosť koreňa $c = -i$ polynómu $f(x) = x^5 - 3x^4 + 2x^3 - 6x^2 + x - 3$.

	1	-3	2	-6	1	-3
$-i$		$-i$	$-1+3i$	$3-i$	$-1+3i$	3
	1	$-i-3$	$1+3i$	$-3-i$	$+3i$	0
$-i$		$-i$	$-2+3i$	$6+i$	$-3i$	
	1	$-2i-3$	$-1+6i$	3	0	
$-i$		$-i$	$-3+3i$	$9+4i$		
	1	$-3i-3$	$-4+9i$	$12+4i \neq 0$	$\implies$	$\boxed{\text{nás.} = 2}$